Solutions – Senior Division

1. (Also J4)

$$\frac{6\times 25}{3\times 5\times 2}=\frac{6\times 25}{6\times 5}=\frac{25}{5}=5,$$

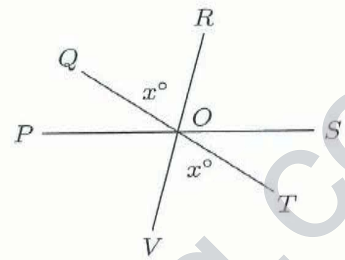
hence (D).

2. (Also I3)

$$\begin{aligned} a &= 2b-5 \\ 2b &= a+5 \\ b &= \frac{a+5}{2}, \end{aligned}$$

hence (D).

3. (Also J15 & I8)
In the diagram, $\angle POR = 120^\circ$ and $\angle QOS = 145^\circ$.



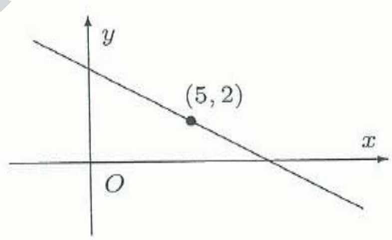
Let $\angle TOV = \angle QOR = x^\circ$.
Then $\angle POR + \angle QOS = \angle POS + x^\circ = 180^\circ + x^\circ$.
So $120 + 145 = 180 + x$ and $x = 85$,

hence (C).

4. $\dfrac{7}{x^2} = 7x^{-2}$,

hence (E).

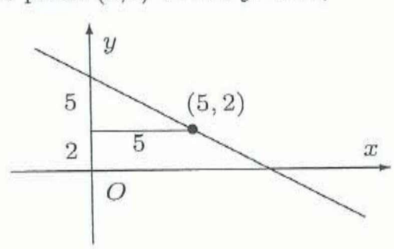
5. In the figure, the line has gradient -1, and it passes through the point (5,2).



(Alternative 1)
Its equation is then $y - 2 = -1(x-5)$ which is $y + x - 7 = 0$.
The y-intercept is when $x = 0$ and is 7,

hence (D).

(Alternative 2)
Draw the perpendicular from the point (5,2) to the y–axis,



and it is easy to see that the y—intercept is 7,

hence (D).

6. (Also I9)
If you begin reading at the top of page 13 and finish at the bottom of page 14 you have read $14-13+1=2$ pages, so if you start at the top of page x and read to the bottom of page y you will have read $y-x+1$ pages,

hence (D).

7. Let the dimensions of the box be x, y and z centimetres.
The volume of the box is then $xyz\ \text{cm}^3$.
We have also that $xy=35$, $yz=60$ and $xz=84$.
So $x^2y^2z^2=35\times60\times84=7\times5\times5\times12\times7\times12$, and then
$xyz=5\times7\times12=420$ and the volume of the box is $420\,\text{cm}^3$,

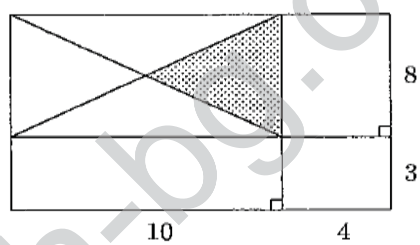
hence (A).

8.

$$\begin{aligned} x &= 3^n+3^n+3^n = 3\times3^n = 3^{n+1}. \\ \text{Hence}\quad x^2 &= \left(3^{n+1}\right)^2 = 3^{2n+2}, \end{aligned}$$

hence (B).

9. (Also I14)
The shaded area is $\frac{1}{4}\times8\times10=20$ square units (quarter of the rectangle).



The total area is 14×11 square units.
The fraction shaded is then $\frac{20}{14\times11}=\frac{10}{7\times11}=\frac{10}{77}$,

hence (E).

10. (Also I15)
The train takes $\frac{1}{4}$ of a minute to pass a post and $\frac{3}{4}$ of a minute to pass through a 600 m tunnel. This means that it takes the front of the train $\frac{1}{2}$ of a minute to pass through the 600 m tunnel and that the train is travelling 0.6 km every $\frac{1}{2}$ minute and so is travelling at $0.6\times120=72\,\text{km/h}$,

hence (D).

11. We can get 2 red balls and 1 white ball in 3 ways, that is by drawing in sequence RRW, RWR or WRR. These ways are mutually exclusive and so the probability of getting 2 R and one W is the sum of the probability of each of these.
Now
$\text{P}(RRW)=\frac{8}{20}\times\frac{7}{19}\times\frac{3}{18}$.
$\text{P}(RWR)=\frac{8}{20}\times\frac{3}{19}\times\frac{7}{18}$.
$\text{P}(WRR)=\frac{3}{20}\times\frac{8}{19}\times\frac{7}{18}$.

The probability of getting 2 red balls and 1 white ball is then

$$3\times\frac{8}{20}\times\frac{7}{19}\times\frac{3}{18}=\frac{7}{5\times19}=\frac{7}{95},$$

hence (E).

12. $16^8 \times 5^{25} = \left(2^4\right)^8 \times 5^{25} = 2^{25} \times 5^{25} \times 2^7 = 2^7 \times 10^{25} = 128 \times 10^{25}$,
so has 3 digits followed by 25 zeros and has 28 digits in total,

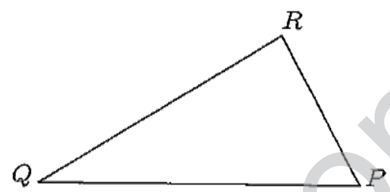
hence (E).

13. We are given $x < y < 0 < z$.
Consider $x + y + z > 0$. $-4 + -3 + 3 = -4 < 0$ so (A) is not always true.
Consider $(x + y)^2 - z > 0$. $(-2 + -1)^2 - 10 = -1 < 0$, so (B) is not always true,
Consider $x + y + z^2 > 0$. $-5 + -1 + 2^2 = -2 < 0$, so (C) is not always true,
Consider $x + y - z > 0$. This is $-ve + -ve + -ve = -ve$, so (D) must be false,
Consider $x + y - z < 0$. This is $-ve + -ve + -ve = -ve$, so must be true,

hence (E).

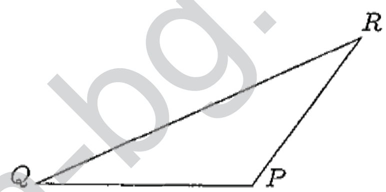
14. Let $0 < A < B < 90°$, with $\sin A = \frac{1}{4}$, $\sin B = \frac{1}{3}$. Then we have 4 cases to consider:

(a) $Q = A$, $P = B$, which is clearly possible.



(b) $Q = 180° - A$, $P = B$ which is not possible since $P + Q = 180° + (B - A) > 180°$.

(c) $Q = A$, $P = 180 - B$ which is possible since $P + Q = 180° + (A - B) < 180°$.



(d) If $Q = 180° - A$ and $P = 180° - B$, there are two obtuse angles, which is impossible.

So $\angle R$ can have two values,

hence (C).

15. (Also I17 & J24)
If a 3-digit number has digits all the same then the number is one of the numbers 111, 222, 333, ..., 999.

Each of these numbers is divisible by 111 and the prime factors of 111 are 3 and 37.

So, one of the two-digit numbers must be a multiple of 3 and the other must be a multiple of 37.

We cannot obtain the products 111, 222 or 333 multiplying two two-digit numbers. We can get
$(1 \times 37) \times (4 \times 3) = 444$,
$(1 \times 37) \times (5 \times 3) = 555$,
$(1 \times 37) \times (6 \times 3) = 666$,
$(1 \times 37) \times (7 \times 3) = 777$,
$(1 \times 37) \times (8 \times 3) = 888$,
$(1 \times 37) \times (9 \times 3) = 999$,
$(2 \times 37) \times (4 \times 3) = 888$,
with 888 being the only one obtainable in two ways, so the number of pairs is 7,

hence (C).

16. (Also I20)
Let the amount of salt in the original mixture be x grams. This means the fraction of salt in the mix is $\frac{x}{450}$. When this saltiness is reduced by 10% by adding y litres of flour, this fraction becomes $\frac{9}{10} \times \frac{x}{450}$ and so

$$\begin{aligned} \frac{x}{450+y} &= \frac{9}{10} \times \frac{x}{450} \\ &= \frac{x}{500}, \end{aligned}$$

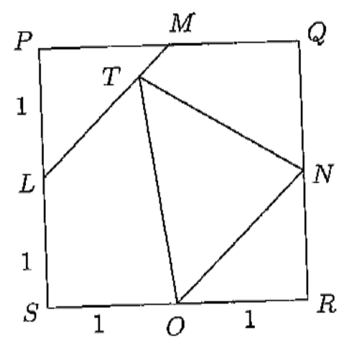
so 50 grams of flour must be added, hence (A).

17. (Also I23)
Let the weights of the bales in kilograms be a, b, c, d, e. Then as all pairs of weights are different, the weights of the five bales are different. Assume then that $a < b < c < d < e$.
The lowest two sums have to be $a+b$ and $a+c$ and the highest two sums have to be $c+e$ and $d+e$.
Also, as each bale is weighed in pairs with four others, the sum of the weights of all pairs must be 4 times their combined weights, so $4(a+b+c+d+e) = 110+112+113+114+115+116+117+118+120+121 = 1156$ and $a+b+c+d+e = 289$.
So we have five equations

$$\begin{aligned} a+b &= 110 & (1) \\ a+c &= 112 & (2) \\ c+e &= 120 & (3) \\ d+e &= 121 & (4) \\ a+b+c+d+e &= 289 & (5) \end{aligned}$$

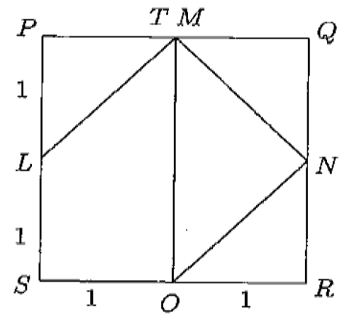
Substituting (1) and (4) in (5) gives $110 + c + 121 = 289$ and so $c = 58$.
Substituting this in (3) gives $e = 62$ so the heaviest bale is 62 kg, hence (E).

18. (Alternative 1)
$ON = \sqrt{2}$ and the height of $\triangle TNO$ is equal to $LO = \sqrt{2}$.



The area of $\triangle TNO$ is then $\frac{1}{2} \times \sqrt{2} \times \sqrt{2} = \frac{1}{2} \times 2 = 1$, hence (B).

(Alternative 2)
Since it does not matter where T lies on LM, let it coincide with M.



The area of $\triangle TON$ is clearly 1,

hence (B).

19.

$$\begin{aligned} 7^{x+1} - 7^{x-1} &= 336\sqrt{7} \\ 7^{x-1}(7^2 - 1) &= 48 \times 7 \times 7^{\frac{1}{2}} \\ 7^{x-1} \times 48 &= 48 \times 7^{1.5} \\ x - 1 &= 1.5 \\ x &= 2.5, \end{aligned}$$

hence (A).

20. (Also J23 & I22)
(Alternative 1)
For each of the three choices to place R in row one, there are two choices to place R in row two and then only one choice in row 3.
So, there are 6 ways of placing the Rs.

R		
		R
	R	

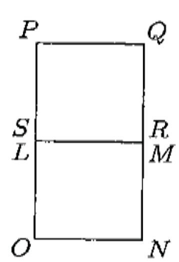
For each of these, there are two choices to place W in row one, then one in row two and one in row three, and for each of these, the placement of B is determined.
Hence the number of different patterns is $6 \times 2 \times 1 = 12$,

hence (D).

(Alternative 2)
There are 6 ways of placing 3 colours in the top row.
Then there are 2 choices for colours in the second row and the final row is then determined, so there are $6 \times 2 = 12$ ways,

hence (D).

21. (Also I24)
P moves along 3 circular arcs. The first rotation about R is 180° with a radius of the diagonal of the square, $\sqrt{2}$. This arc is then of length $\pi \times \sqrt{2}$.

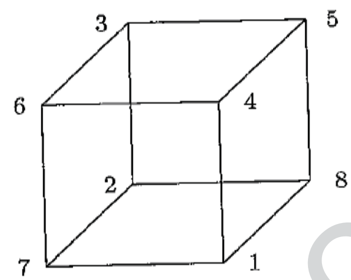


The next rotation about Q is $180°$ with radius 1, so the length is π.
The third rotation is about P so P does not move.
The last rotation is $180°$ about L with radius 1 so has length π.
The total length of the path traced out is then

$$\pi \times \sqrt{2} + \pi + \pi = \pi(2 + \sqrt{2}),$$

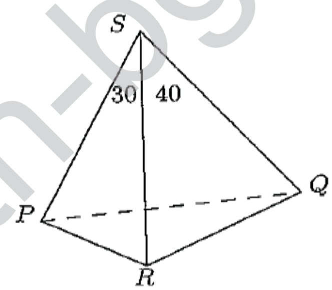
hence (A).

22. (Also J29 & I25)
There are 6 faces and hence 6 face sums. Since each vertex lies on 3 faces, the sum of all the face sums of the cube is $3(1+2+3+4+5+6+7+8) = 108$, so if there are 6 equal face sums, that sum must be $108 \div 6 = 18$.



The figure gives an example of 6 equal face sums, which is the maximum number possible, hence (E).

23. Consider the tetrahedron PRQS as shown.



Clearly, $\angle PSQ < \angle PSR + \angle QSR$. Hence $\angle PSQ < 70°$.
Also $\angle QSR < \angle PSR + \angle PSQ$. Hence $\angle PSQ > 10°$.
Therefore the possible sizes of $\angle PSQ$ are $11°, 12°, \ldots, 69°$. So there are 59 possible sizes, hence (B).

24. (Alternative 1)
If x is a solution then

$$\begin{aligned} a + x &= a^2 - 2a\sqrt{a-x} + a - x \\ 2a\sqrt{a-x} &= a^2 - 2x \\ 4a^2(a-x) &= a^4 - 4a^2x + 4x^2 \\ 4x^2 &= a^3(4-a). \end{aligned}$$

Consequently $0 \le a \le 4$.
The possible positive integer solutions are $1, 2, 3, 4$.
Check: $a = 1$: $\sqrt{1+x} + \sqrt{1-x} > 1$, not a solution;
$a = 2, x = 2$ a solution;
$a = 4, x = 0$ a solution.

What about $a = 3$?

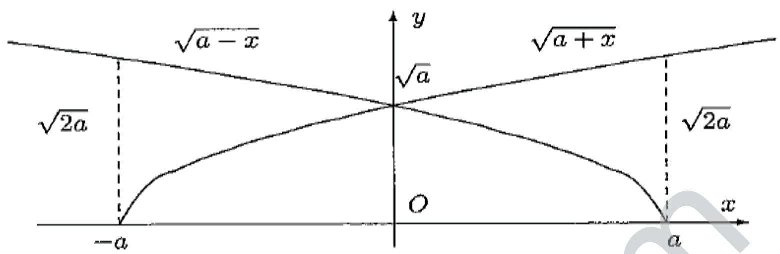
$$\sqrt{3+0}+\sqrt{3-0} = 2\sqrt{3} > 3 > \sqrt{3+3}+\sqrt{3-3} = \sqrt{6}$$

so, by continuity, there is a solution for x between 0 and 3.

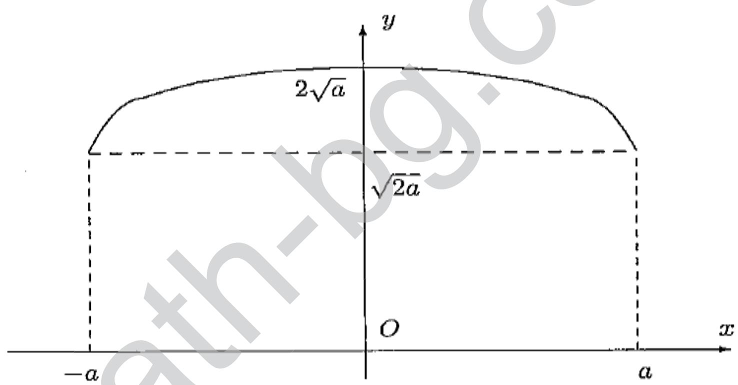
There are then 3 values of a:- 2, 3 and 4,

hence (D).

(Alternative 2)
Sketch the graphs of $\sqrt{a+x}$ and $\sqrt{a-x}$ together.



Adding these two, we have a function only defined between $-a$ and a.



It is clear that the equation $\sqrt{a+x}+\sqrt{a-x} = a$ can only have a solution when $\sqrt{2a} \le a \le 2\sqrt{a}$. Since $a > 0$ is given, we may square throughout with impunity to get $2a \le a^2 \le 4a$ and so $2 \le a \le 4$, giving three possibilities, 2, 3 and 4,

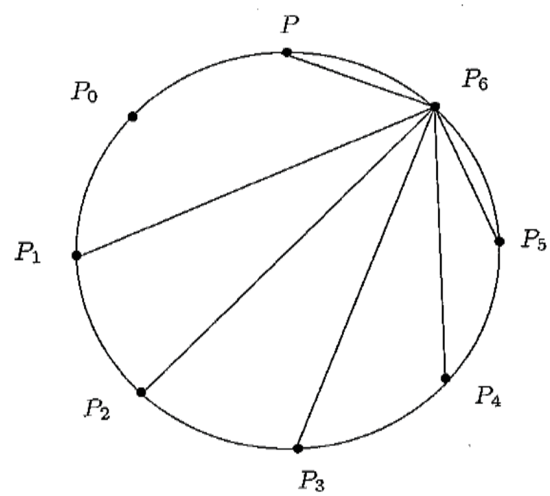
hence (D).

25. We are given 8 points on a circle, one of them P, where all points other than P lie on a different number of chords joining these points. There are two cases:

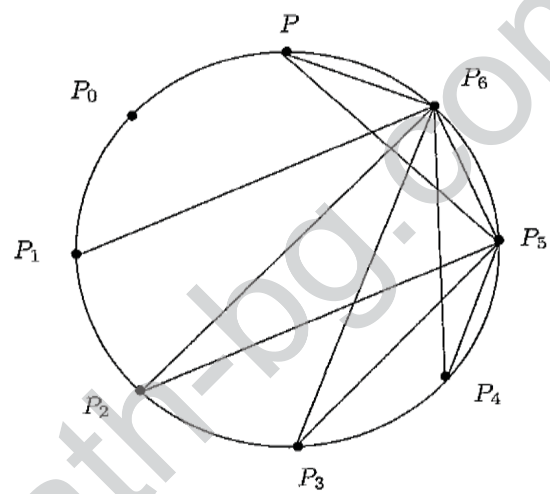
(1) One point does not lie on any chord. This means that the points other than P must lie on 0, 1, 2, 3, 4, 5 and 6 chords. Label these points P_0, P_1, P_2, P_3, P_4, P_5 and P_6 respectively.

(2) One point is connected to every other point. This means that the points other than P must lie on 1, 2, 3, 4, 5, 6 and 7 chords. Label these points P_1, P_2, P_3, P_4, P_5, P_6 and P_7, respectively.

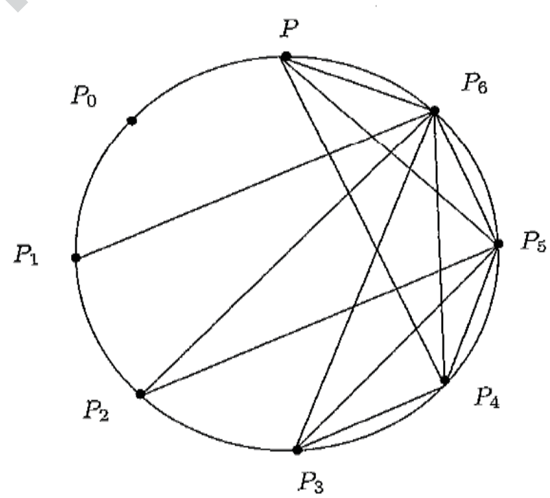
Case 1.
It does not matter in what order the points lie on the circle, so arrange them as shown and since P_6 is connected to every point other than P_0 we get the following diagram:



Now connect P_5. It cannot be connected to P_1 (already connected to P_6), so must connect to every other point other than P_0.

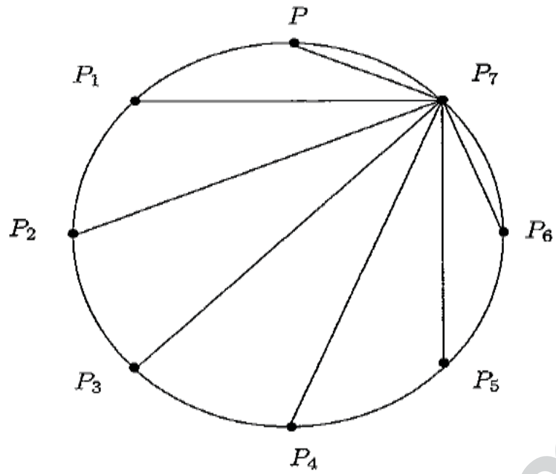


Now, noting that P_0, P_1, P_2, P_5 and P_6 are on their limit, P_4 can only connect to P_3 and P and this completes the diagram as follows:

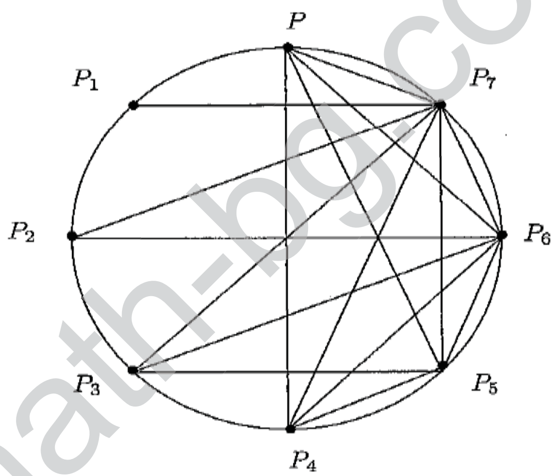


This case shows that P must lie on 3 chords.

Case 2.
As before, it does not matter in what order the points lie on the circle, so arrange them as shown and since P_7 is connected to every point we get the following diagram:



Now, continuing with P_6 and so on in a similar fashion to Case 1, we obtain the following completed diagram:



In this case P lies on 4 chords.

The minimum number of chords on which P lies is then 3 from Case 1,

hence (C).

26. (Also I27 & J27)
There are 18 possible digit sums for two-digit numbers: 1, 2, 3, ..., 17, 18. There is only one two-digit number with digit sum 1, namely 10; and there is only one two-digit number with digit sum 18, that is 99. Therefore the largest number of two-digit numbers that can be written on the whiteboard, such that no three numbers have the same digit sum, is $1 \times 2 + 2 \times 16 = 34$. Thus the smallest number of students in the class is 35.

27. (Alternative 1)
We are given

$$\begin{aligned} a+b+c &= 4 \qquad (1)\\ a^2+b^2+c^2 &= 10 \qquad (2)\\ a^3+b^3+c^3 &= 22 \qquad (3) \end{aligned}$$

Squaring (1) and subtracting (2) we get $2(ab+bc+ca) = 6$ and $ab+bc+ca = 3$.

So

$$\begin{aligned} a^3+b^3+c^3-3abc &= (a+b+c)(a^2+b^2+c^2-ab-bc-cb) \\ 22-3abc &= 4(10-3) \\ abc &= -2 \end{aligned}$$

So, using the result that for a cubic equation $x^3+px^2+qx+r=0$ with roots $a,b,c,$ that $a+b+c=-p$, $ab+bc+ca=q$ and $abc=-r$, tells us that a, b and c are roots of the equation

$$\begin{aligned} x^3-4x^2+3x+2 &= 0 \\ (x-2)(x^2-2x-1) &= 0. \\ x &= 2, 1\pm\sqrt{2} \end{aligned}$$

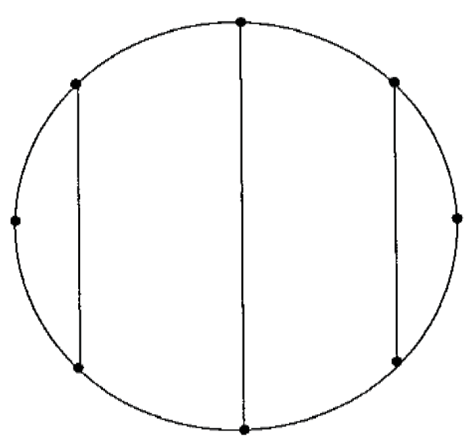
Now $$(1+\sqrt{2})^4 = 1+4\sqrt{2}+6\times 2+4\times 2\sqrt{2}+4$$
$$(1-\sqrt{2})^4 = 1-4\sqrt{2}+6\times 2-4\times 2\sqrt{2}+4$$
$$2^4 = 16$$
Then $$\begin{aligned} a^4+b^4+c^4 &= 16+2(1+12+4) \\ &= 50. \end{aligned}$$

(Alternative 2)

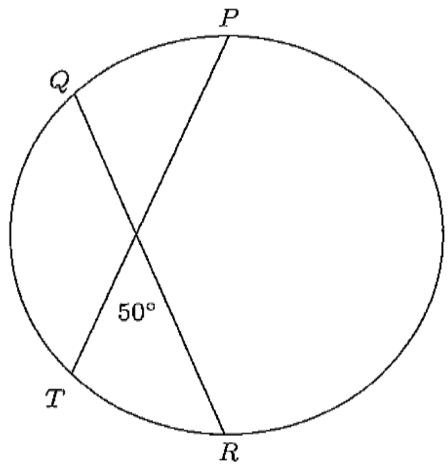
$$\begin{aligned} (a^2+b^2+c^2)^2 &= a^4+b^4+c^4+2(a^2b^2+b^2c^2+c^2a^2) \\ &= 100 \\ (ab+bc+ca)^2 &= a^2b^2+b^2c^2+c^2a^2+2abc(a+b+c) \\ 9 &= a^2b^2+b^2c^2+c^2a^2-2\times 2\times 4 \\ a^2b^2+b^2c^2+c^2a^2 &= 9+16 = 25 \\ a^4+b^4+c^4 &= 100-2\times 25 = 50. \end{aligned}$$

Comment
This solution does not show that there exist real numbers a, b and c which satisfy the given conditions, but it would seem reasonable to assume in the context of the question that they do.

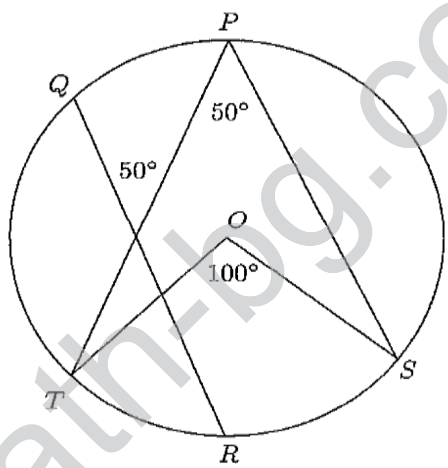
28. (Also I29)
(Alternative 1)
The question does not stipulate that the two diagonals must share a common vertex. However, for each diagonal in a regular polygon with more than 5 sides, there are parallel diagonals through other vertices. An example follows:



Consider the case where a pair of diagonals PT and QR intersect at an angle of $50°$ as in the following diagram.



Then, if arc $PQ <$ arc PT, then there must exist a point S on arc PT such that $PS \parallel QR$. So TS subtends an angle of $50°$ at the circumference of the circle and also an angle of $100°$ at the centre.

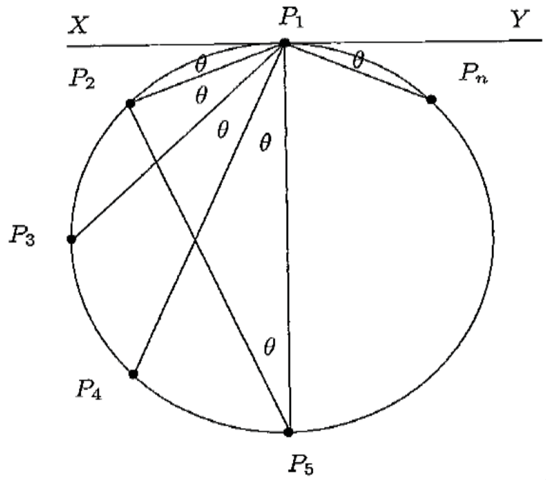


It follows that the angle subtended at the centre of the circle by a single side of the polygon must be a divisor of $100°$ and also of $360°$. The highest common factor of 100 and 360 is 20, so the smallest number of edges the polygon can have is $\frac{360}{20} = 18$.

(Alternative 2)
For the angle between two diagonals in a regular $n-$gon to be $50°$, there must be two vertices of this polygon such that they divide the perimeter of the polygon in ratio 50 : 130.
Hence there exists a postive integer a such that $\frac{a}{n-a} = \frac{5}{13}$.
Therefore $13a = 5n - 5a$, and $18a = 5n$. Since 5 and 18 are relatively prime, n must be divisible by 18. So $n \geq 18$ and the example of $n = 18$, $a = 5$ show that $n = 18$ is possible.

Hence the smallest value of n is 18.

(Alternative 3)
Inscribe the polygon $P_1, P_2, P_3, \ldots$ in a circle. Let XY be the tangent at P.



Angle $P_2P_1P_3$, angle $P_3P_1P_4$ and so on are equal (θ) as they are subtended by equal arcs at P_1.

Angle XP_1P_2 and angle YP_1P_n are also equal (to θ) by the alternate segment theorem.

Hence $\angle XP_1Y$ is divided into n equal angles by edges P_1P_2, P_1P_n and by all diagonals drawn from P_1 to the other $n-1$ vertices of the polygon.

Each of these n angles is equal to $\dfrac{180^\circ}{n}$.

All angles between diagonals are integer multiples of this basic angle as moving from P_1 to P_2 rotates all diagonals by $\frac{360}{n} = 2 \times \frac{180}{n}$, so that each diagonal is parallel to a diagonal, edge or tangent at P_1 and this applies to each vertex.

So, to generate 50° between diagonals, we need to find the smallest value of n which makes $\frac{180}{n}$ a factor of 50.

The highest common factor of 180 and 50 is 10, so n is minimum when $\frac{180}{n} = 10$ and this is when $n = 18$.

29. (Also I30)
We are looking for a maximum product

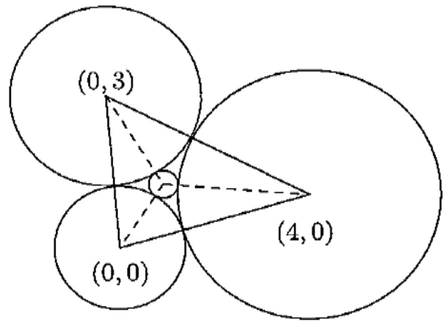
$$n_1 \times n_2 \times n_3 \times \ldots \times n_k$$

where $n_1 + n_2 + n_3 + \cdots + n_k = 19$.

If any factor n_i is ≥ 5, it can be replaced by two factors 2 and $n_i - 2$ which leave the sum unchanged, but increases the product since $2 \times (x-2) > x$ for $x \geq 5$. So, every factor in the largest product is ≤ 4.

Similarly, if any factor n_i is equal to 4, it can be replaced by 2×2 with no change to the product, so we shall do this and then every factor is ≤ 3.
If any factor is 1, it can be combined with another factor, replacing $1 \times n_i$ by $(n_i + 1)$ which increases the product, so now all factors in the largest product are 2 or 3.
If there are three or more 2s, $2 \times 2 \times 2$ can be replaced by 3×3 to increase the product. So in the largest product, there are at most two 2s.
There is only one way that 19 can be written as such a sum: there are five 3s and two 2s.
So the maximum product is $3^5 \times 2^2 = 972$.

30. Joining the centres of the three main circles, we see that we have a 3-4-5 right-angled triangle. Let us give its vertices coordinates as shown. Let us also say that the small circle in the middle has radius r and its centre has coordinates (x, y).



Now, looking at the lengths of the three dotted lines joining the outer vertices to the centre of the small circle, we have the equations

$$x^2 + y^2 = (1+r)^2 \quad (1)$$
$$x^2 + (3-y)^2 = (2+r)^2 \quad (2)$$
$$(4-x)^2 + y^2 = (3+r)^2. \quad (3)$$

Subtracting the first one from each of the others gives

$$(3-y)^2 - y^2 = (2+r)^2 - (1+r)^2$$
$$(4-x)^2 - x^2 = (3+r)^2 - (1+r)^2,$$

which simplify to $3y = 3 - r, \qquad 2x = 2 - r$.
Substituting from these back into (1) gives

$$\left(1 - \frac{r}{2}\right)^2 + \left(1 - \frac{r}{3}\right)^2 = (1+r)^2$$

which, upon multiplying by 36 and simplifying, gives

$$23r^2 + 132r - 36 = 0.$$

Applying the usual formula, we have

$$r = \frac{-66 + \sqrt{5184}}{23} = \frac{-66+72}{23} = \frac{6}{23} = \frac{p}{q}$$

(the other root is negative, which is impossible).
So, $p + q = 6 + 23 = 29$.