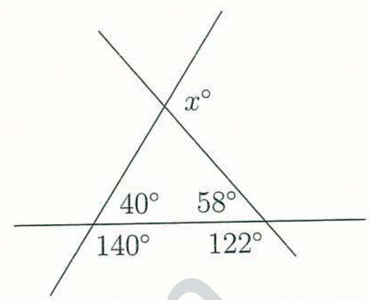
Solutions – Senior Division

1. $(2000+9)-(2000-9)=2000+9-2000+9=18$,

hence (E).

2. (Also I2)

As we have straight lines, we can find the two additional angles as shown. Then angle x° is the exterior angle of the triangle and so $x=40+58=98$,



hence (E).

3. As $y=kx$ passes through $(-2,-1)$, so $-1=-2k$, $k=\dfrac{1}{2}$,

hence (D).

4. $(0.6)^{-2}=\left(\dfrac{6}{10}\right)^{-2}=\left(\dfrac{3}{5}\right)^{-2}=1\Big/\left(\dfrac{3}{5}\right)^{2}=\dfrac{25}{9}$,

hence (D).

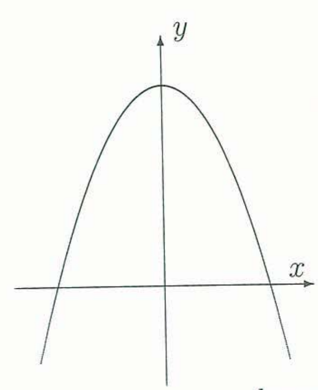
5. (Also I7)
$(x-y)-2(y-z)+3(z-x)=x-y-2y+2z+3z-3x=-2x-3y+5z$,
hence (A).

6. The cost of the beads, in dollars, is
$1+2+3+4+5+\cdots+13+\cdots+5+4+3+2+1=2\times 1+2\times 2+\cdots+2\times 12+13=169$,
hence (B).

7. $1*(2*3)=1*(2+\frac{1}{3})=1*(\frac{7}{3})=1+\frac{3}{7}=\frac{10}{7}$,

hence (B).

8. The graph is symmetric about the y-axis so $b=0$.
The parabola is convex up, so $a<0$.
The y-intercept is positive, so $c>0$.
Since we do not know the relative values of a and c, (C) and (D) may or may not be true, but (B) must be true since $a<0$ and $c>0$,



hence (B).

9. (Also I11)
There are 1000 students with 570 girls, so there are 430 boys.
One-quarter of the students travel by bus so 250 students travel by bus.
313 boys do not travel by bus so $430 - 313 = 117$ boys do take the bus.
So, the number of girls who travel by bus is $250 - 117 = 133$,

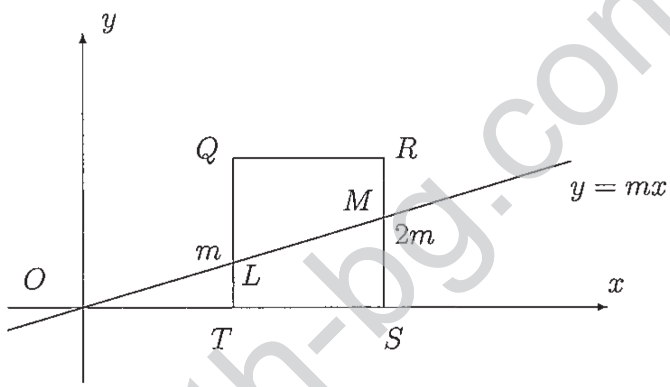
hence (E).

10. The number of ways of drawing out two green caps is $\binom{6}{2}$, the number of ways of drawing out two red caps is $\binom{3}{2}$ and the number of ways of drawing out two caps is $\binom{9}{2}$. So

$$\text{P(two the same)} = \frac{\binom{6}{2} + \binom{3}{2}}{\binom{9}{2}} = \frac{15+3}{36} = \frac{1}{2},$$

hence (A).

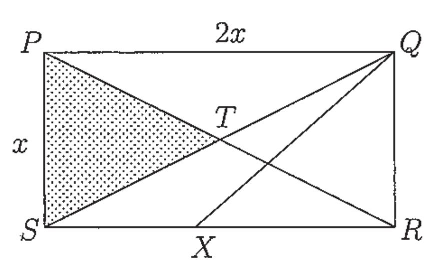
11. Let the equation of the line be $y = mx$ and let the line intersect QT at L and RS at M. Then the y-coordinate of L is m and the y-coordinate of M is $2m$.



As LM bisects the area of the square $QRST$ then the area of trapezium $LTSM$ is equal to the area of trapezium $LQRM$. Now $LT = m$, $QL = 1 - m$, $SM = 2m$, $RM = 1 - 2m$ and $TS = QR = 1$.
So, $\frac{1}{2}(m + 2m) \times 1 = \frac{1}{2}(1 - m + 1 - 2m) \times 1$, $6m = 2$ and $m = \frac{1}{3}$. The equation of the line is $y = \frac{1}{3}x$,

hence (B).

12. The area of the pentagon $PQRST$ is equal to the area of the rectangle $-$ area $\triangle PTS = 2x^2 - \frac{1}{2}x \times x = \frac{3}{2}x^2$.



So, the area of each half of the pentagon is $\frac{3}{4}x^2$ and the area of $\triangle QRX = \frac{3}{4}x^2$.
So $\frac{1}{2}RX \times x = \frac{3}{4}x^2$ and $RX = \frac{3}{2}x$,

hence (D).

13. Given $5^x - 5^{x-2} = 120\sqrt{5}$, let $s = 5^x$. Then $5^{x-2} = \dfrac{s}{25}$ and

$$\begin{aligned} s - \frac{s}{25} &= 120\sqrt{5} \\ 24s &= 25 \times 120\sqrt{5} \\ s &= 125\sqrt{5} = 5^{\frac{7}{2}}. \end{aligned}$$

So $x = \dfrac{a}{b} = \dfrac{7}{2}$ and $a + b = 9$,

hence (D).

14. The circle $x^2 + y^2 = 50$ has radius $\sqrt{50} \approx 7.07$, so for each $x = -7, -6, \ldots, 7$, there are two points with this x-coordinate. Hence there are 30 points with an integral x-coordinate. Similarly, there are 30 points with an integral y-coordinate. However, there are 12 points where both coordinates are integers, $(\pm1, \pm7), (\pm5, \pm5)(\pm7, \pm1)$. So there are $60 - 12 = 48$ points with at least one integral coordinate,

hence (C).

15. (Also I17)
Any arrangement in which 4 and 5 in some order are in the second and fourth positions, $x4x5x$ and $x5x4x$, is automatically an eyebrow. There are 2 ways of placing the 4 and 5, and then there are 6 ways of placing the other 3 numbers in the three positions, giving $2 \times 6 = 12$ eyebrows of this type.
However, it is also possible that 4 is at one end next to the 5, with 3 one place from the other end. There are 2 ways of choosing the end for the 4, and then the remaining two digits 1 and 2 can be placed in two ways, so there are $2 \times 2 = 4$ eyebrows of this type.
The total number of eyebrows is $12 + 4 = 16$,

hence (A).

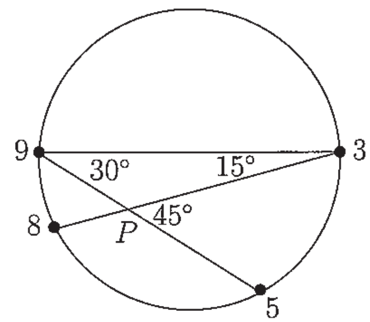
16. The degree-4 equation $(x^2 - x)^2 = 18(x^2 - x) - 72$ has at most 4 real solutions.
Let $x^2 - x = m$. Then

$$\begin{aligned} m^2 - 18m + 72 &= 0 \\ (m-6)(m-12) &= 0 \\ m &= 6 \text{ or } 12. \end{aligned}$$

If $m = 6$, then $x^2 - x = 6$, $x^2 - x - 6 = 0$, $(x-3)(x+2) = 0$ and $x = 3$ or -2.
If $m = 12$, then $x^2 - x = 12$, $x^2 - x - 12 = 0$, $(x-4)(x+3) = 0$ and $x = 4$ or -3.
The sum of the positive solutions is $3 + 4 = 7$,

hence (B).

17. The arc joining 5 and 3 is $\frac{1}{6}$ of the circumference of the circle, so subtends an angle of 30° at the point 9 (angle at centre twice angle at circumference). Similarly, the arc joining the points 8 and 9 is $\frac{1}{12}$ of the circumference of the circle, so subtends an angle of 15° at the point 3.
Let the line joining 9 and 5 intersect the line joining 3 and 8 at the point P.



Then $\angle 3P5$ is the exterior angle of $\triangle 3P9$ and so is $30° + 15° = 45°$,

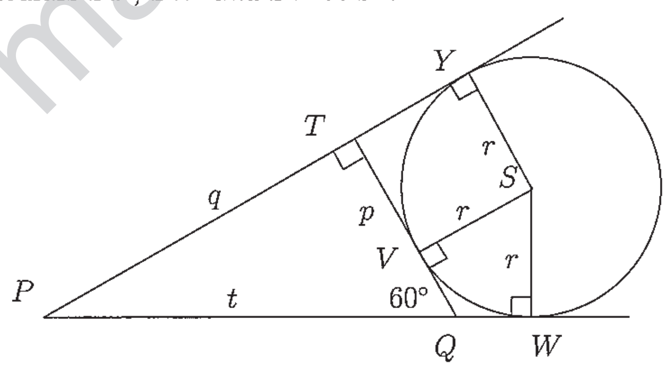
hence (D).

18. Let the positive fraction be $\dfrac{a}{b}$ where a and b have no common factors. Then $\dfrac{x}{60} = \dfrac{a}{b} + \dfrac{b}{a} = \dfrac{a^2+b^2}{ab}$ and, since a and b have no common factors, ab and a^2+b^2 have no common factors and $ab = 60$.

So we have $a^2 + b^2 = x$ and $ab = 60 = 3 \times 4 \times 5$, where a, b are coprime. So ab can be 1×60, 3×20, 4×15 or 5×12.
So there are 4 possible values of x: $1^2 + 60^2 = 3601$, $3^2 + 20^2 = 409$, $4^2 + 15^2 = 241$ and $5^2 + 12^2 = 169$,

hence (D).

19. Let r be the radius of the circle. With the notation shown, $t = 10$ and $p = 5$. Draw the perpendiculars SY, SW and SV as shown.



Since $t = 2p$ and $\angle PQT = 60°$, then $\angle PTQ = 90°$.
So $q = \sqrt{75} = 5\sqrt{3}$.
Area $\triangle PQT = \frac{1}{2} tp \sin \angle PQT = \frac{1}{2} \times 10 \times 5 \times \dfrac{\sqrt{3}}{2} = \dfrac{25\sqrt{3}}{2}$.

$$\begin{aligned}\text{Area } \triangle PQT &= \text{area } \triangle PTS + \text{area } \triangle PQS - \text{area } \triangle QTS \\ &= \tfrac{1}{2}PT.YS + \tfrac{1}{2}PQ.WS - \tfrac{1}{2}QT.VS \\ &= \tfrac{1}{2}qr + \tfrac{1}{2}tr - \tfrac{1}{2}pr = \tfrac{1}{2}r(q+t-p).\end{aligned}$$

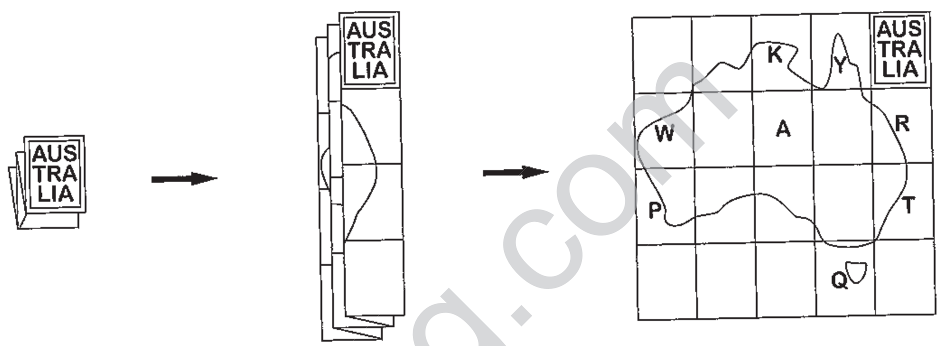
Then $\dfrac{25\sqrt{3}}{2} = \dfrac{1}{2}r(5\sqrt{3}+10-5)$ and $25\sqrt{3} = r(5+5\sqrt{3})$.

So $r = \dfrac{25\sqrt{3}}{5+5\sqrt{3}} = \dfrac{25\sqrt{3}}{5(1+\sqrt{3})} = \dfrac{5\sqrt{3}(\sqrt{3}-1)}{(1+\sqrt{3})(\sqrt{3}-1)} = \dfrac{5(3-\sqrt{3})}{3-1} = \dfrac{5(3-\sqrt{3})}{2}$,

hence (B).

20. (Also J23 & I21)

After refolding along vertical folds, the four panels are stacked, from top to bottom, YK, RAW, TP, Q.



After folding the horizontal folds, the second and fourth will be reversed giving YK, WAR, TP, Q,

hence (E).

21. (Also J25 & I22)

Any palindromic number $xyyx$ can be written as $1000x + 100y + 10y + x = 1001x + 110y$, where x and y are integers and $1 \leq x \leq 9$ and $0 \leq y \leq 9$.

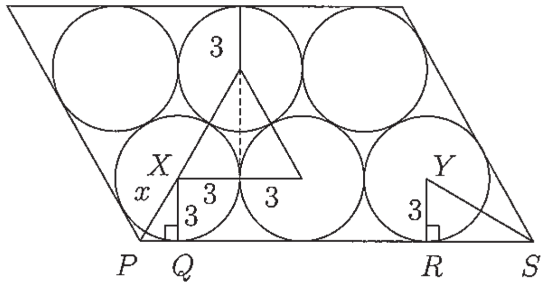
Now $1001 = 7 \times 143$, so 1001 and every multiple of it is divisible by 7. There are nine such multiples 1001, 2002, 3003, …, 9009.

110 is not divisible by 7, so $110y$ is not divisible by 7 unless y is divisible by 7, and this occurs when $y = 0$ (already dealt with above) or $y = 7$. This gives another nine palindromes, 1771, 2772, …, 9779.

So there are $9 + 9 = 18$ such palindromes,

hence (D).

22. The area of the parallelogram is equal to the base times the height.



The height of the parallelogram is 6 plus the altitude of the equilateral triangle of side 6 units which is $6+3\sqrt{3}$. Hence triangle XPQ is a $30°, 60°, 90°$ triangle and $PQ=\sqrt{3}$.

Similarly triangle SYR is a $30°, 60°, 90°$ triangle and $RS=3\sqrt{3}$.

So $PS = 12+4\sqrt{3}$ and the area of the parallelogram, in square centimetres, is

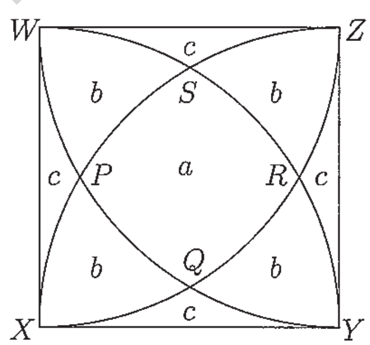
$$(6+3\sqrt{3})(12+4\sqrt{3}) = 12(2+\sqrt{3})(3+\sqrt{3}) = 12(9+5\sqrt{3}),$$

hence (D).

23. (Also I24)
A is either loyal or a traitor.
Assume that A is loyal. B's claim shows B is loyal, F's claim shows F is loyal, D's claim shows D is a traitor. A's claim shows E is a traitor. C's claim is now impossible as B and F are loyal; C would either claim both were loyal or both were traitors.
Hence A is a traitor. So by F's claim F is a traitor. A says E is a traitor so E is loyal. Hence by E's claim, D is a traitor. By D's claim C is a traitor. Finally by C's claim B is loyal.
So, B and E are loyal and A, C, D and F are traitors,

hence (D).

24. *Alternative 1*
We wish to calculate the area of the region $PQRS$.



The square is divided into 9 regions: $PQRS$ with area a, PWS and 3 others with area b and WSZ and 3 others with area c.

Hence $\qquad a + 4b + 4c = 1 \qquad (1)$

The quarter circle $WSRYX$ has area $\frac{\pi}{4}$, so

$$a + 3b + 2c = \frac{\pi}{4} \qquad (2)$$

Consider the region $XPSRY$. $\angle SXY = \frac{\pi}{3}$ so sector SXY has area $\frac{\pi}{6}$ as does sector SYX. The triangle SXY is equilateral, so its area is $\frac{\sqrt{3}}{4}$. Then

$$a + 2b + c = \frac{\pi}{3} - \frac{\sqrt{3}}{4} \qquad (3)$$

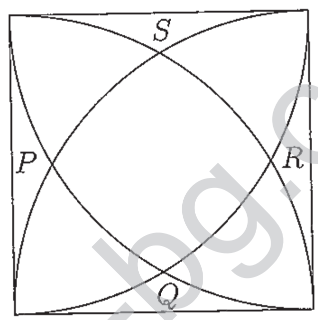
$$(2) - (3) \Rightarrow \qquad b + c = \frac{\sqrt{3}}{4} - \frac{\pi}{12} \qquad (4)$$

$$(1) - 4 \times (4) \Rightarrow \qquad a = 1 - \sqrt{3} + \frac{\pi}{3},$$

hence (C).

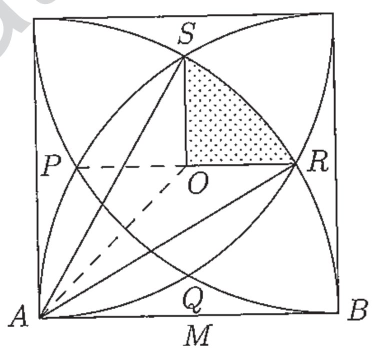
Alternative 2

We wish to calculate the area of the region $PQRS$.



Let O be the centre of the square.

Consider the area OSR which is one-quarter of the required area.



$\triangle ASB$ is equilateral and $\triangle AOM$ is a right-angled isosceles triangle, so $\angle SAO = 60° - 45° = 15°$ and $\angle RAS = 30° = \frac{\pi}{6}$.

So, area of sector $SAR = \frac{1}{2}r^2\theta = \frac{\pi}{12}$.

Area $\triangle OAR = \frac{1}{2}OR \times OM = \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) \times \frac{1}{2} = \frac{1}{8}(\sqrt{3} - 1)$ and then the area of the region $ORS = \dfrac{\pi}{12} - \dfrac{\sqrt{3}-1}{4}$.

The required area is $4 \times \left(\dfrac{\pi}{12} - \dfrac{\sqrt{3}-1}{4}\right) = \dfrac{\pi}{3} - \sqrt{3} + 1$,

hence (C).

25. We have, for all positive integral n,

$$\begin{aligned} f_n(x) &= f(f_{n-1}(x)) \\ &= \frac{f_{n-1}(x)+6}{f_{n-1}(x)} = x \\ \text{Then} \quad f_{n-1}(x)+6 &= xf_{n-1}(x) \\ f_{n-1}(x) &= \frac{6}{x-1} = x. \\ \text{So} \quad x^2 - x - 6 &= 0 \\ (x-3)(x-2) &= 0 \\ x = 3 \quad \text{and} \quad x = -2. \end{aligned}$$

So there are 2 solutions,

hence (A).

26. Let the four positive integers be x, $2x$, $3x$ and y.
Then

$$\begin{aligned} \frac{1}{x} + \frac{1}{2x} + \frac{1}{3x} + \frac{1}{y} &= \frac{19}{20} \\ \frac{1}{y} &= \frac{19}{20} - \frac{1}{x} - \frac{1}{2x} - \frac{1}{3x} \\ &= \frac{19}{20} - \frac{11}{6x} = \frac{57x - 110}{60x} \\ y &= \frac{60x}{57x - 110}. \end{aligned}$$

Consider $x = 1$, y neither integral nor positive.
Consider $x = 2$, $y = 30$. Works.
Consider $x = 3$, $y = \dfrac{180}{61}$ not integral.
Consider $x = 4$, $y = \dfrac{240}{118}$ not integral.
Consider $x \geq 5$, $y \leq \dfrac{180}{117} = \dfrac{60}{39} < 2$, so there are no other solutions. The four numbers are then 2, 4, 6 and 30 with sum 42.

27. (Also J28 & I27)
The seven smallest ascending 3-digit numbers are
$n = 123, 124, 125, 126, 127, 128$ and 129.

$6 \times 123 = 738 \quad 6 \times 127 = 762$
$6 \times 124 = 744 \quad 6 \times 128 = 768$
$6 \times 125 = 750 \quad 6 \times 129 = 774$
$6 \times 126 = 756$

In none of these cases is $6n$ an ascending number.

Now, $6n$ must end with a 0, 2, 4, 6 or 8 and the sum of its digits must be divisible by 3.
124, 134 and 234 are the only ascending 3-digit numbers ending with 4 and $6n = 744, 804, 1404$ in these cases, none ascending.
Hence n ends in a 6 or an 8.

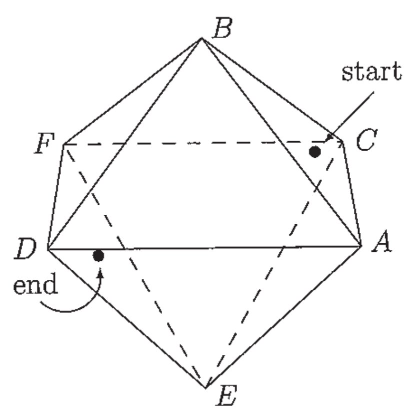
Consider those ending in 6.

n	$6n$	n	$6n$
136	816	146	876
156	936	236	1416
246	1476	256	1536
346	2076	356	2136
456	2736		

None of the $6n$ are ascending.

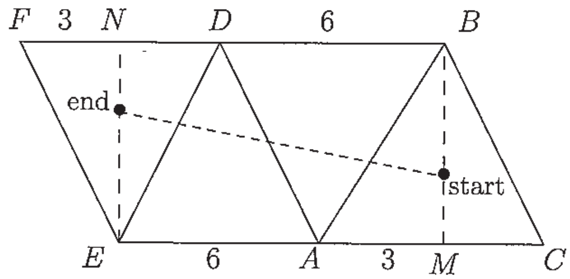
Consider then n ending with an 8, so is $ab8$ with $a < b < 8$. Then the carry into the tens digit of the product $6 \times n$ is 4. The next largest digit in n we can consider is $b = 7$ which gives $42 + 4$, so we get 6 in the tens digit of $6n$ and a carry of 4 to the hundreds digit of $6n$. This means the units digit of $6a$, must be less than 1 (otherwise $6n$ would not be ascending). Hence the first digit of n is 5, $6n$ is 3468 and n is 578.

28. The path must cross one of the edges AB, AC or BC. By symmetry, we may assume it crosses AB into face ABD.

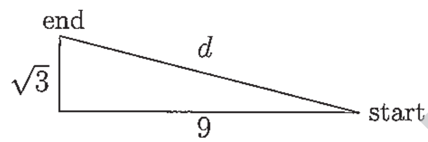


Where next? The path must cross one of AD or BD. By symmetry again, suppose it crosses AD into face ADE. Then it must cross DE into the final face.

Now, lay out flat these four faces.



The altitudes BM and EN are of length $3\sqrt{3}$ and the start and end points trisect them.
We then have a right-angled triangle with sides 9 and $\sqrt{3}$ as shown.



The length of the hypotenuse of this triangle is d, the length of the path.
So, by Pythagoras,

$$d^2 = (\sqrt{3})^2 + 9^2 = 3 + 81 = 84.$$

29. (Also I30)
Alternative 1
Without any loss of generality, assume the track is circular, and put in the stations C_1 to C_5 with C_1 at the top. There are then five $72°$ gaps.

Then add the B stations. As there are 4 of these, there must remain a clear gap between two C stations. Assume that this is the C_1C_5 gap. Put B_1 in the C_1C_2 gap $x°$ from C_1.

Then $x \leq 18$ or B_4 would lie in the C_1C_5 gap and through symmetry, would put the clear gap elsewhere.

Then place the A stations. Clearly, A_1 must go in the $72°$ C_1C_5 gap. Let it make an angle $y°$ with C_1.

Then, if $y < 24$, A_3 lies between B_4 and C_4, and using the fact that the angles between the C stations are $72°$, the angles between the B stations are $90°$ and the angles between the A stations are $120°$, we can fill in all the angles as shown in figure 1.

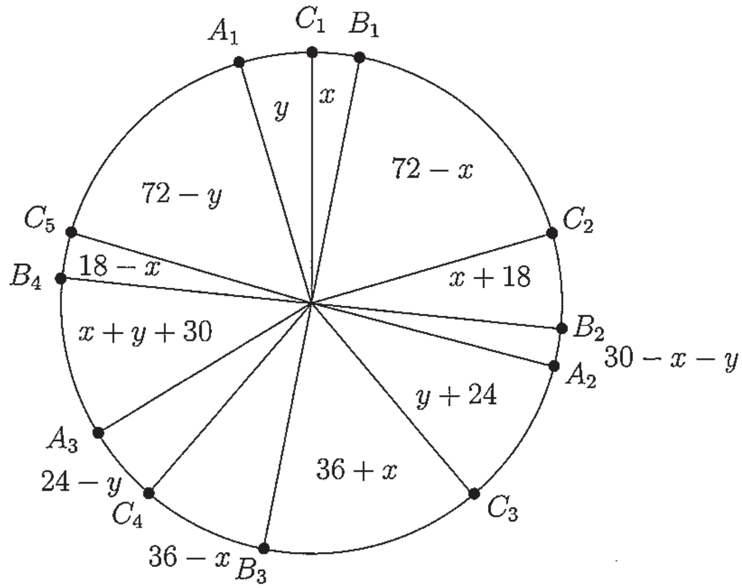


figure 1

If, however, from $\angle A_2OB_2$, $x + y > 30$, then A_2 is between B_2 and C_2 and we obtain the angles as in figure 2.

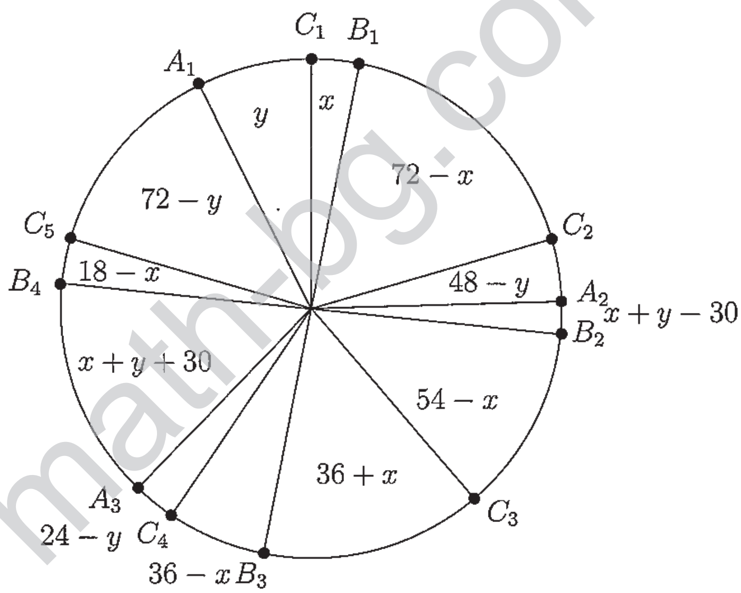


figure 2

In both cases, it is clear that the largest angles are $72 - x$, $72 - y$ and $x + y + 30$. So, the smallest maximum will be when all three are equal:

$$72 - y = 72 - x = x + y + 30$$

This gives $x = y = 14$ and the smallest of the largest gaps is $58°$.

For $24 \le y \le 48$, then A_3 is between C_4 and B_3 as in figure 3.

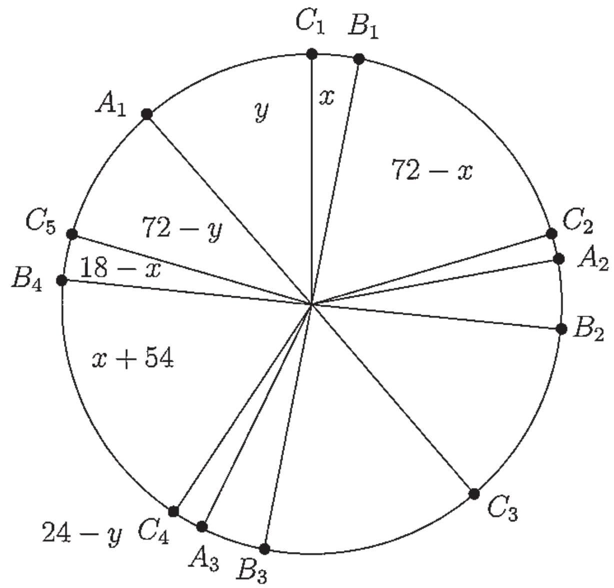


figure 3

This forces two gaps of $x + 54$ and $72 - x$, where one or both must be greater than 58.

When $48 < y < 72$, then A_2 lies between B_1 and C_2 and A_3 lies between B_3 and C_3 as in figure 4.

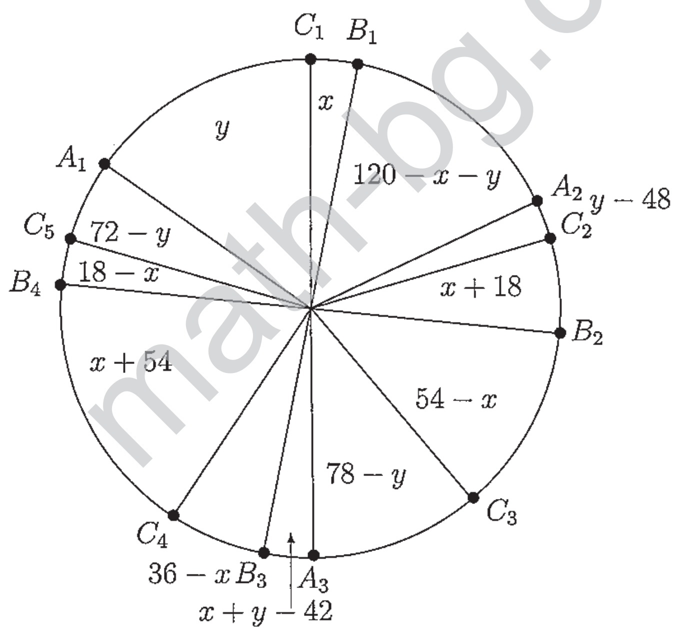


figure 4

Then the largest angles are y, $120 - x - y$ and $x + 54$.
Then, when $y = 120 - x - y = x + 54$, we get $y = 58$, $x = 4$ and the smallest of the largest angles is $y = 54 + x = 120 - x - y = 58$, the same as for figure 1.

The minimum of the largest distances between stations is then
$\dfrac{58}{360} \times 1080 = 174\,\text{km}$.

Alternative 2

The maximum spacing must be at least 90 km, as there are 12 stations in all. However, the station spacing is severely constrained, so the actual maximum will be well in excess of this.

The maximum spacing must be at most 216 km, as this is the spacing of the C stations. Adding in the B stations will not reduce this, as there will always be a pair of C stations with no B station in between.

First, a 'reasonably good' solution:
Fix the C stations as a reference set, and position the A and B stations relative to them. Initially, align the B and C stations at km 0; the largest gap is 216 km, at either side of km 0. Next, align the A and B stations at km 540; the other A stations will both be in the previous longest sections, reducing them to 180 km each. The pattern and spacings are:

$$BC\ 180\ A\ 36\ C\ 54\ B\ 162\ C\ 108\ AB\ 108\ C\ 162\ B\ 54\ C\ 36\ A\ 180\ BC.$$

This is already close to optimum, as we have 2 gaps of 180 km and 2 of 162 km.

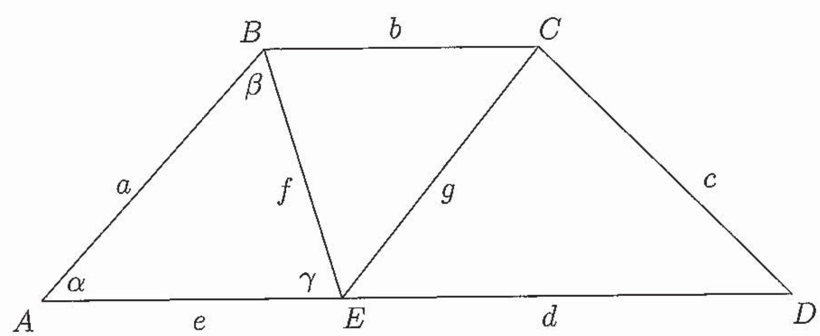
Now, improve the fit by breaking the symmetry: alter the A and B starting points to decrease both of the 180 km gaps. Move the A system forward by x km and the B system forward by $2x$ km, reducing both 180 km gaps by x km each. As the second of the 162 km gaps is now increased by $2x$ km, x cannot exceed 6. Using $x = 6$, the new pattern is

$$C\ 12\ B\ 174\ A\ 30\ C\ 66\ B\ 150\ C\ 114\ A\ 6\ B\ 96\ C\ 174\ B\ 42\ C\ 42\ A\ 174\ C.$$

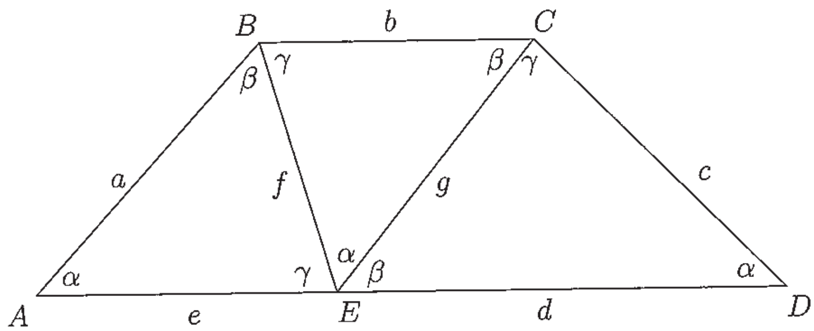
Note that we now have 3 gaps of 174 km each and the next largest is 150 km. Further alterations will result in at least one of the 174 km gaps increasing in length, so we have an optimum: 174 km.

30. *Alternative 1*

Label the edges and angles of the leftmost triangle as follows:



We have that neither $ABCE$ nor $BCDE$ is a parallelogram, so we can fill in the angles as follows:



The ratios in the triangles are

$$a : e : f = g : f : b = d : c : g = p : q : r$$

where p, q, r is a Pythagorean triple (in some order) with no common factors ($\gcd(p, q) = \gcd(p, r) = \gcd(q, r) = 1$, using the fact that in a Pythagorean triple, a common factor of two of the numbers is also common to the third, so all common factors can be eliminated from the ratio).

Then

$$\frac{d}{e} = \frac{d}{g} \cdot \frac{g}{f} \cdot \frac{f}{e} = \frac{p}{r} \cdot \frac{p}{q} \cdot \frac{r}{q} = \frac{p^2}{q^2}$$

so that

$$2009 = d + e = \frac{p^2}{q^2} \cdot e + e = \frac{p^2 + q^2}{q^2} \cdot e \quad \Rightarrow \quad 2009q^2 = (p^2 + q^2)e.$$

Then $(p^2 + q^2)|2009 = 7^2.41$, so $p^2 + q^2 \in \{7, 41, 49, 287, 2009\}$.

Inspection suggests that $p^2 + q^2 = 4^2 + 5^2 = 41$. To see that this is the only possibility, suppose $p^2 + q^2$ were a multiple of 7. However, modulo 7, a square is equivalent to 0, 1, 2 or 4, so $p^2 \equiv q^2 \equiv 0 \pmod 7 \Rightarrow 7|\gcd(p, q) = 1$.

Thus either $p = 4$ and $q = 5$ or vice versa, and in either case $r = 3$ to get a Pythgorean triple.

So, either $a : e : f = g : f : b = d : c : g = p : q : r = 4 : 5 : 3$ or
$a : e : f = g : f : b = d : c : g = p : q : r = 5 : 4 : 3$.

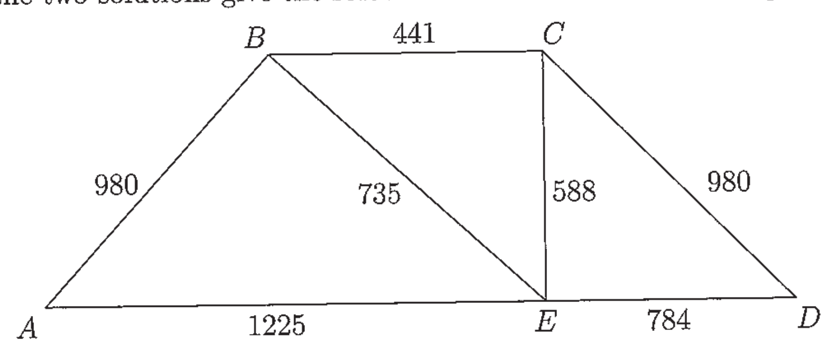
In either case,

$$b = \frac{b}{f} \cdot \frac{f}{e} \cdot \frac{e}{2009} \cdot 2009 = \frac{r}{q} \cdot \frac{r}{q} \cdot \frac{q^2}{p^2 + q^2} \cdot 2009 = \frac{r^2}{p^2 + q^2} \cdot 2009 = \frac{9 \times 2009}{41} = 441.$$

So, the length of BC is 441 units.

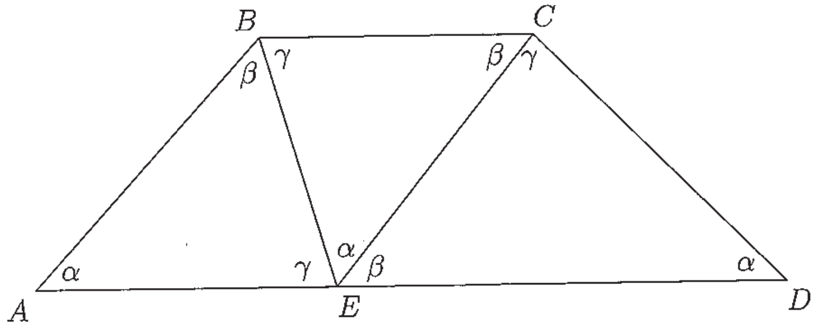
Comment

In fact, the two solutions give the solution below and its mirror image.

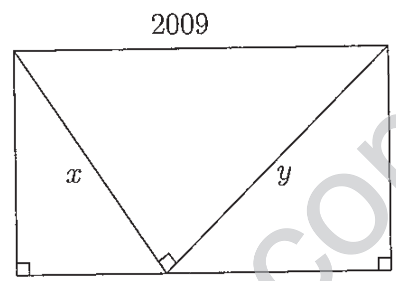


Alternative 2

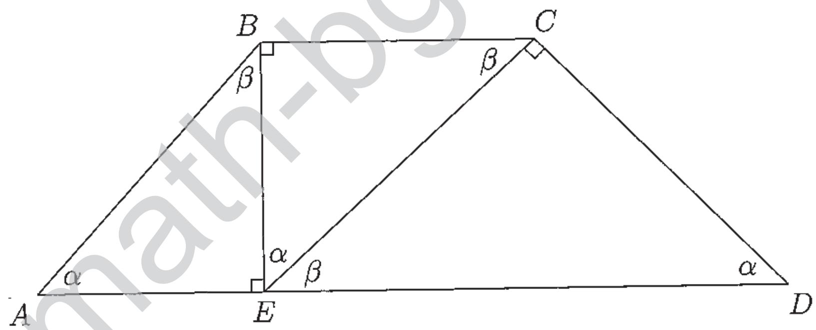
Let $\angle BAE = \alpha$, $\angle ABE = \beta$ and $\angle AEB = \gamma$. Then, as neither $ABCE$ nor $BCDE$ is a parallelogram, we can fill in all the angles as shown.



Then either $\alpha = 90$ or $\beta = 90$ ($\gamma = 90$ is a reflection of $\beta = 90$).
If $\alpha = 90$, then we get the case

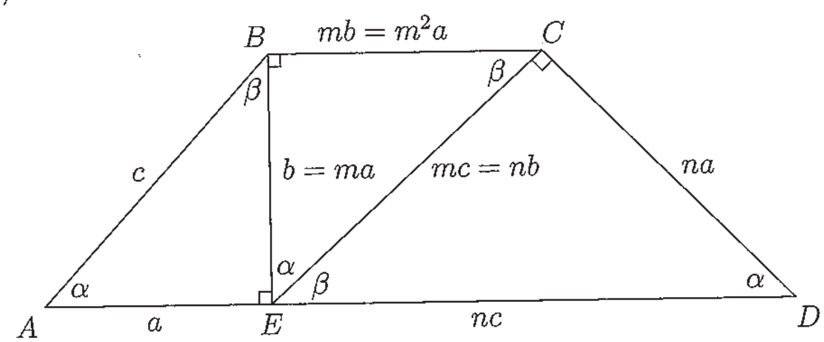


where $x^2 + y^2 = 2009$, which has no integer solutions, so $\alpha \neq 90$ and $\gamma = 90$.
So, we have



where $\triangle AEB$ |||$\triangle EBC$ |||$\triangle DCE$, (where ||| means is similar to).

So, from the diagram and the similar triangles, there is a rational number m such that $EB = ma$, $BC = mb$ and $CE = mc$.
Similarly, there is a rational number n such that $CD = na$, $CE = nb$ and $DE = nc$.



We have

$$\begin{aligned} a + nc &= 2009 \\ a + n\left(\frac{nb}{a}\right) &= 2009 \\ a + n^2 a &= 2009 \end{aligned}$$

Now n is rational, so let $n = \dfrac{p}{q}$ with p and q having no common factors.

We now have the (Diophantine) equation

$$\begin{aligned} a\left(1 + \frac{p^2}{q^2}\right) &= 2009 \\ a(p^2 + q^2) &= 2009q^2 \end{aligned}$$

Since p and q have no common factors, we have $q^2|a$ and hence $(p^2 + q^2)|2009 = 7^2 \times 41$.

So, as in Alternative 1, we have $p^2 + q^2 = 41$ and $p = 4$ and $q = 5$ or $p = 5$ and $q = 4$.

This means that $n = \dfrac{4}{5}$ or $\dfrac{5}{4}$.

As $EC > BE$ we have $n = \dfrac{5}{4}$.

Then

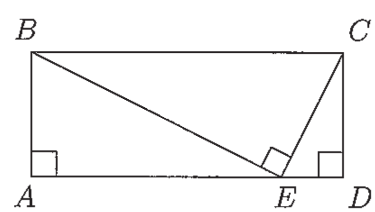
$$\begin{aligned} a = \frac{2009}{1+n^2} &= \frac{16 \times 2009}{41} = 16 \times 49 = 784 \\ nc &= 2009 - a \\ c = \frac{4}{5}(2009 - 784) &= \frac{4 \times 1225}{5} = 980 \\ b^2 = c^2 - a^2 &= (c + a)(c - a) = 1764 \times 196 = 42^2 \times 14^2 \\ b &= 588 \end{aligned}$$

So, $m = \dfrac{b}{a} = \dfrac{588}{784} = \dfrac{3}{4}$ and $BC = mb = \dfrac{3}{4} \times 588 = 441$.

Alternative 3

There are 2 possible arrangements that satisfy the conditions: either D is a right angle or it is not. Note that $2009 = 7^2 \times 41$.

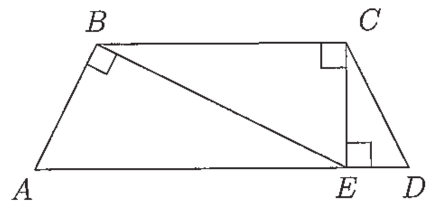
Case 1:



The similar triangles are EDC, CEB and BAE. The side lengths form Pythagorean triples, multiples of an irreducible set (a, b, c). The simplest format is $ED, DC, CE = ka^2, kab, kac$; $CE, EB, BC = kac, kbc, kc^2$; and $BA, AE, EB = kab, kb^2, kbc$. We require that $AD = k(a^2 + b^2) = 2009$. Then $k = 1$ and $a^2 + b^2 = c^2 = 2009$, or

$k = 7^2$ and $a^2 + b^2 = c^2 = 41$.
Neither case is possible, as neither 41 nor 2009 is a perfect square.

Case 2:



The similar triangles are EDC, CEB and BAE. The side lengths form Pythagorean triples, multiples of an irreducible set (a, b, c).
The simplest format is $ED, DC, CE = ka^2, kac, kab$; $CE, EB, BC = kab, kbc, kb^2$; and $BA, AE, EB = kac, kc^2, kbc$.
We require that $AD = k(a^2 + c^2) = 2009$, and hence $k < 45$. Then $a^2 + c^2 = 1, 7, 41, 49, 287$ or 2009.
The available irreducible sets are (3,4,5), (5,12,13), (8,15,17), (7,24,25), (20,21,29), (12,35,37) and (9,40,41), where a is either of the first 2 values in the set.
In the $a^2 + c^2 = 41$ case, we have a solution for $(a, b, c) = (3, 4, 5)$, scaled up by 49 times.
The other 5 cases all fail.

Hence $BC = 49 \times 3^2 = 441$.