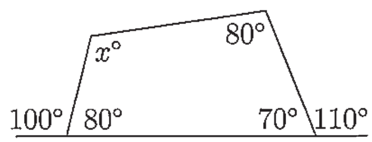
1. $8002 - 2008 = 5994$,

hence (E).

2. The difference between $\frac{1}{20}$ and $\frac{2}{10}$ is $\frac{2}{10} - \frac{1}{20} = \frac{4-1}{20} = \frac{3}{20}$,

hence (E).

3. Using supplementary angles and the angle sum of a quadrilateral, we get



$x = 360 - 80 - 80 - 70 = 130$,

hence (D).

4. (Also J6 & I4)

$$\frac{200 \times 8}{200 \div 8} = \frac{200 \times 8 \times 8}{200} = 8 \times 8 = 64,$$

hence (D).

5. *Alternative 1*
As $x^2 - 4x + 3 = (x-3)(x-1)$, the minimum value occurs when $x = 2$ and is $1 \times -1 = -1$,

hence (A).

Alternative 2
$x^2 - 4x + 3 = (x-2)^2 - 1$, so the minimum value is -1,

hence (A).

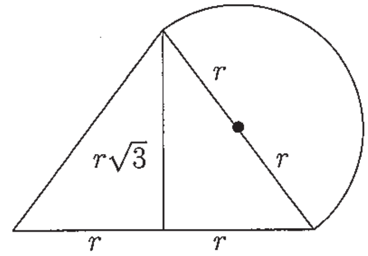
6. When the money is divided the two shares are \$1.75 and \$1.25.
The ratio of the larger to smaller is $175 : 125 = 7 : 5$,

hence (B).

7. (Also I10)
When 1000^{2008} is written as a numeral, it consists of the digit 1 followed by 3×2008 zeros, so there are $1 + 3 \times 2008 = 6025$ digits,

hence (C).

8. Let the radius of the circle be r. Then the base of the triangle is $2r$ and, by Pythagoras, the height of the triangle is $r\sqrt{3}$.
The area of the semicircle is $\dfrac{\pi r^2}{2}$.



The area of the triangle is $\dfrac{2r \times r\sqrt{3}}{2} = r^2\sqrt{3}$.
So, ratio of the area of the semicircle to that of the triangle is

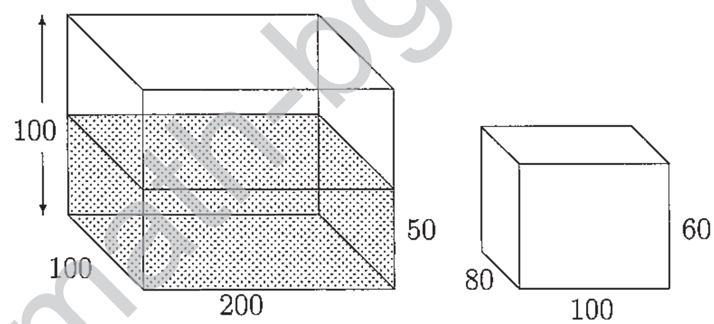
$$\frac{\pi r^2}{2} : r^2\sqrt{3} = \pi : 2\sqrt{3},$$

hence (B).

9. As $\cos x = 0.5 = \dfrac{1}{2}$, $\cos^2 x = \dfrac{1}{4}$ and so $\sin x = \dfrac{\sqrt{3}}{2}$. Then $\tan x = \dfrac{\sqrt{3}}{2} \Big/ \dfrac{1}{2} = \sqrt{3}$.
So $\tan x = \sqrt{3}$ is the largest,

hence (E).

10. (Also J24 & I15)
The volume of the water is $100 \times 200 \times 50 = 1\,000\,000$ cm^3.



The volume of the prism is $80 \times 100 \times 60 = 480\,000$ cm^3.
So, when the prism is placed in the tank, the new height of water is

$$\frac{1480000}{20000} = 74 \text{ cm.}$$

The prism is 60 cm high so is covered by 14 centimetres of water,

hence (B).

11. Now $2^{500} = 32^{100}$, $3^{400} = 81^{100}$, $4^{300} = 64^{100}$, $5^{200} = 25^{100}$ and $6^{100} = 6^{100}$.
The largest is $81^{100} = 3^{400}$,

hence (B).

12. In this problem, the distribution is symmetric, unimodal and the centre point is a valid point and so the most likely value and expected value are the same.
The expected outcome is

$$1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = \frac{21}{6} = \frac{7}{2}.$$

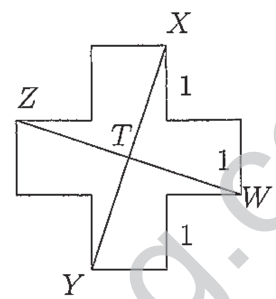
So, in 100 throws, the expected score would be $100 \times \frac{7}{2} = 350$,

hence (D).

13. (Also I16)
Factorising, we get $2008 = 2^3 \times 251$ where 251 is prime. So, to complete to a square, we must multiply by $2 \times 251 = 502$,

hence (D).

14. Draw in the ZW line as shown.



The point where XY and ZW intersect is T and, by symmetry, XT, WT, YT and ZT are the same length.
Since the sides of the rectangle are XY and XT, the ratio is $2:1$,

hence (C).

15. Given $f(x) = ax^2 + bx + c$, we have

$$\begin{aligned} f(1) &= a + b + c = 2 && (1) \\ f(2) &= 4a + 2b + c = 3 && (2) \\ f(3) &= 9a + 3b + c = 1 && (3) \end{aligned}$$

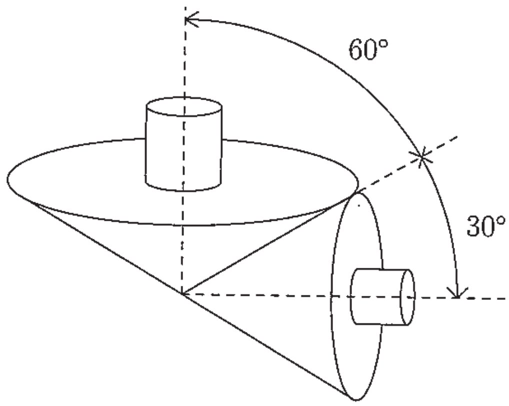
which gives

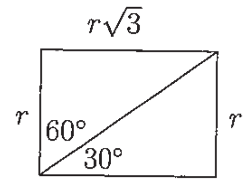
$$\begin{aligned} 3a + b &= 1 && (2) - (1) && (4) \\ 5a + b &= -2 && (3) - (2) && (5) \\ 2a &= -3 && (5) - (4) && (6) \end{aligned}$$

Thus $a = -\frac{3}{2}$, $b = \frac{11}{2}$ and $c = -2$,

hence (A).

16. Let the radius of the smaller roller be r.





Then the radius of the larger roller is $r\sqrt{3}$.

So, the circumference of the larger roller is $2\pi r\sqrt{3}$ and that of the smaller roller is $2\pi r$.

So, when the larger roller makes 1 revolution, the smaller roller makes

$2\pi r\sqrt{3}\,/\,2\pi r = \sqrt{3}$ revolutions,

hence (D).

17. Consider the sets with 1, 2 and 3 elements.
1 element: 1, 2, 3, 4, 5, 6 giving 6
2 elements: 1, 3; 1, 4; 1, 5; 1, 6; 2, 4; 2, 5; 2, 6; 3, 5; 3, 6 and 4, 6 giving 10
3 elements 1, 3, 5; 1, 3, 6; 1, 4, 6 and 2, 4, 6 giving 4.
Any set of 4 elements must contain consecutive numbers.

So there are 20 subsets in all,

hence (C).

18. (Also J23 & I21)
The amount of water collected is proportional to the areas of the two roofs. So

volume collected on farmhouse roof : volume collected on barn roof

is $200 : 80 = 5 : 2$.

So, if Farmer Taylor is to collect as much water as possible, the empty space in the tanks has to be in the same ratio $5 : 2$.

Currently, there is $100 - 35 = 65\,\text{kL}$ available in the farmhouse tank and $25 - 13 = 12\,\text{kL}$ in the barn tank.

Now, the ratio $65 : 12$ is greater than $5 : 2$, so we must pump some water from the barn tank into the house tank.

If we pump x kL from the barn tank into the house tank, then the empty space in the house tank is $65 - x$ kL and the empty space in the barn tank is $12 + x$ kL.

So, we want $65 - x : 12 + x = 5 : 2$, which gives $\dfrac{65-x}{5} = \dfrac{12+x}{2}$, $130 - 2x = 60 + 5x$, $7x = 70$ and $x = 10$.

So, to collect the maximum amount of water possible we must pump 10 kL from the barn tank into the farmhouse tank,

hence (D).

19. More generally, suppose that

$$u_1 = a,\ u_2 = b,\ u_n = u_{n-1} - u_{n-2}\ (n \geq 3),$$

where a and b are fixed real numbers.

Then

$$\begin{array}{ll} u_3 = b - a & u_4 = b - a - b = -a \\ u_5 = -a - (b - a) = -b & u_6 = -b - (-a) = a - b \\ u_7 = a - b - (-b) = a & u_8 = a - (a - b) = b \end{array}$$

and the sequence repeats with period 6.

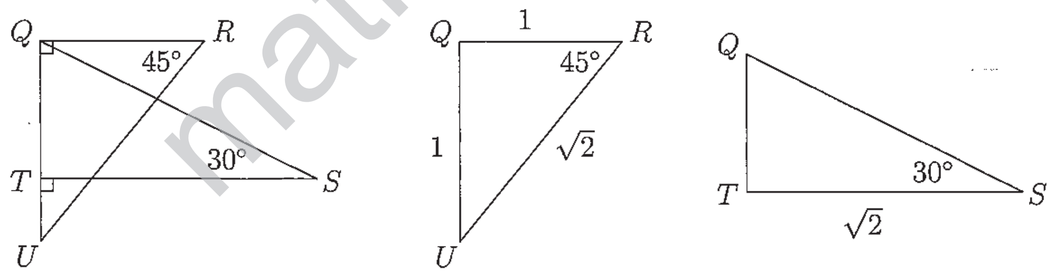
Since $2008 \equiv 4 \pmod 6$, $u_{2008} = u_4 = -a$.

So, in this case, $u_{2008} = u_4 = -\sqrt{2}$,

hence (A).

20. Let $QR = 1$, then $RU = ST = \sqrt{2}$.

From the right-angled triangle QTS, we get $\dfrac{QT}{\sqrt{2}} = \tan 30^\circ = \dfrac{1}{\sqrt{3}}$.



Therefore the area of $\triangle QST = \dfrac{1}{2} \times \sqrt{2} \times \dfrac{\sqrt{2}}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}$.

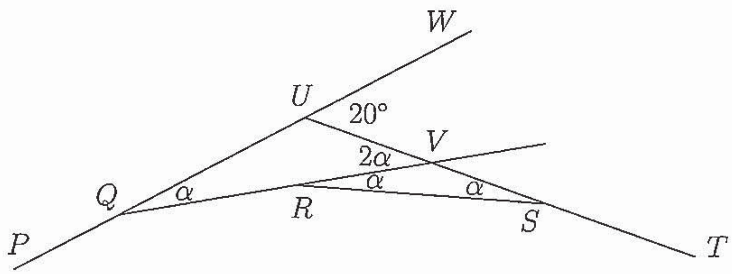
The area of $\triangle QRU = \dfrac{1}{2} \times 1 \times 1 = \dfrac{1}{2}$.

So, the ratio of area $\triangle QRU$: area $\triangle QST = \dfrac{1}{2} : \dfrac{1}{\sqrt{3}} = \sqrt{3} : 2$,

hence (D).

21. *Alternative 1*

The equal sides PQ, QR, RS and ST are as shown. Then the sides PQ and TS extended meet at U at an angle of $20°$.



Let α be the exterior angle of the polygon.
Then $\angle QVU = 2\alpha$ (exterior angle of a triangle) and $\angle VUW = 3\alpha = 20°$.
As

$$3\alpha = 20°, \qquad \alpha = \frac{20}{3} = \frac{360}{54},$$

so the polygon has 54 sides,

hence (E).

Alternative 2

In the regular polygon, the sides PQ and TS meet at U where $\angle QUS = 160°$.
Consider the regular polygon as being generated by rotating a side n times.

Three steps rotate the side $20°$, so $18 \times 3 = 54$ steps return the side to its original position. So, there are 54 sides,

hence (E).

22. If $a^3 \leq 2008$ then $a \leq 12$, as $12^3 = 1728 < 2008 < 2197 = 13^3$.

We have to count the numbers divisible by $2^3, 3^3, 4^3, 5^3, 6^3, 7^3, 8^3, 9^3, 10^3, 11^3$ and 12^3.

Now $\dfrac{2008}{8} = 251$, so there are 251 numbers divisible by 2^3. This count also includes numbers divisible by $4^3, 6^3, 8^3, 10^3$ and 12^3.

The notation $\lfloor x \rfloor$ means the integer part of x.

There are $\left\lfloor \dfrac{2008}{27} \right\rfloor = 74$ numbers divisible by 3^3. This includes numbers divisible by $6^3, 9^3$ and 12^3.

There are $\left\lfloor \dfrac{2008}{125} \right\rfloor = 16$ numbers divisible by 5^3 which include those divisible by 10^3.

There are $\left\lfloor \dfrac{2008}{343} \right\rfloor = 5$ numbers divisible by 7^3 and 1 number divisible by 11^3.

But we have counted some numbers twice, those divisible by $6^3, 10^3$ and 12^3, so we must subtract $\left\lfloor \dfrac{2008}{216} \right\rfloor = 9$ and $\left\lfloor \dfrac{2008}{1000} \right\rfloor = 2$. (Those divisible by 12^3 have been counted in these.)

So the number of cubic factors is $251 + 74 + 16 + 5 + 1 - 9 - 2 = 336$,

hence (B).

23. *Alternative 1*

Suppose a is a 3-digit palindrome, b is a 3-digit palindrome and $a-b$ is a 3-digit palindrome. Then, with x, y, u and v all single digit positive numbers,

$$\begin{aligned} a &= 100x + 10y + x \\ b &= 100u + 10v + u \\ a-b &= 100(x-u) + 10(y-v) + (x-u) \end{aligned}$$

Then it follows that $x-u>0$ and $y-v \geq 0$.
So, we need the number of pairs (x,u) with $9 \geq x > u > 0$ times the number of pairs (y,v) with $9 \geq y \geq v \geq 0$.

The first number is $\binom{9}{2} = 36$, since $1 \leq x \leq 9$, $1 \leq u \leq 9$ and $x > u$.

The second number is $\binom{10}{2} + 10 = 45 + 10 = 55$, since $y = v$ or $0 \leq y \leq 9$, $0 \leq v \leq 9$ and $y > v$.

So, the total number is $36 \times 55 = 1980$,

hence (B).

Alternative 2

Let the numbers be xyx and uvu, where x, y, u and v are single-digit positive numbers.
Now, for $xyx - uvu$ to be a palindrome which is 3-digit, $x > u$ and $x \leq 9$ and $u \geq 1$, while at the same time $y \geq v$ and $y \leq 9$ and $v \geq 0$.

If $x = 9$ there are 8 possibilities for u.
If $x = 8$ there are 7 possibilities for u.
$\vdots$
If $x = 2$ there is 1 possibility for u.
So, there are $8 + 7 + 6 + \ldots + 2 + 1 = 36$ possible values for x and u.

If $y = 9$, there are 10 possibilities for v (0 to 9).
If $y = 8$, there are 9 possibilities for v (0 to 8).
$\vdots$
If $y = 1$, there are 2 possibilities for v (0 and 1).
If $y = 0$, v must be 0, so there is one possible value for v.
So, there are $10 + 9 + 8 + \ldots + 1 = 55$ values of y and v.

So, the total number of pairs is $36 \times 55 = 1980$,

hence (B).

Alternative 3

The valid pairs have the form $a = xyx$ with $2 \leq x \leq 9$ and $0 \leq y \leq 9$, and with $b = uvu$ where $1 \leq u \leq x-1$ and $0 \leq v \leq y$.

This gives $(x-1)(y-1)$ pairs with $a = xyx$.

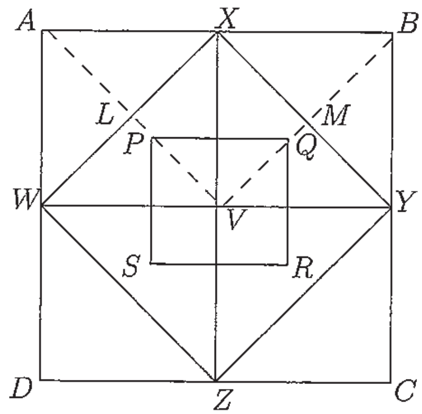
So, for each x in the range 2 to 9 there are 55 choices for y making

$$(1 + 2 + 3 \cdots + 8) \times 55 = 36 \times 55 = 1980$$

pairs in all,

hence (B).

24. Consider the view of the figure through one face of the cube (which is a projection of all the points and lines onto this face of the cube).



$ABCD$ is one face of the larger cube.
X, Y, Z, W and V are five of the six vertices of the octahedron.
PQ is an edge of the smaller cube. (This corresponds to a projection of all points onto the face $ABCD$. However, the projected PQ has the same length as the original PQ as PQ is parallel to the face $ABCD$.)

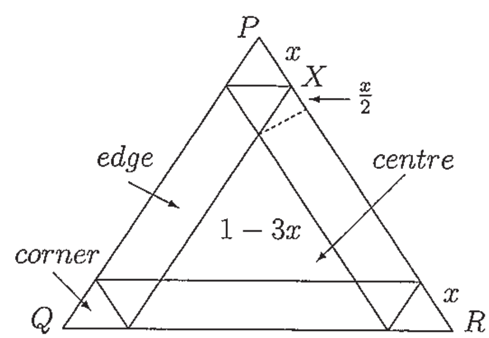
P is the centroid (centre) of the $\triangle VXW$, hence $PV : LP = 2 : 1$.

Also $AL = LV$, so $PV : AV = 2 : (2 + 1 + 3) = 1 : 3$.

This means that $PQ : AB = 1 : 3$,

hence (D).

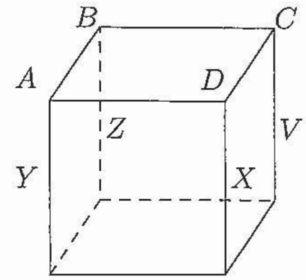
25. There are only three different areas, corner, edge and centre. Clearly the edge area is bigger than the corner area. So we are trying to maximise the smaller of the corner area and the central area.
If $PX = x$, then the two areas are an equilateral triangle of side x and another of side $1 - 3x$.

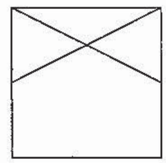


When these two areas are equal $x = 1 - 3x$ and so $PX = \dfrac{1}{4}$,

hence (C).

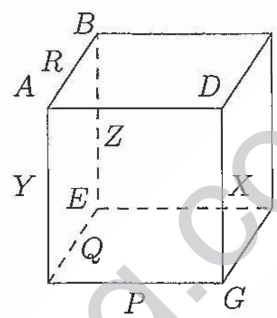
26. Assume the cube has an edge length 1. X, Y, V and Z are mid-points of edges.
Four lines AV, BX, DZ and CY, from the top face, produce 1 internal point inside the top half of the cube, which from symmetry, will be $\frac{1}{4}$ of a unit from the centre of the face.

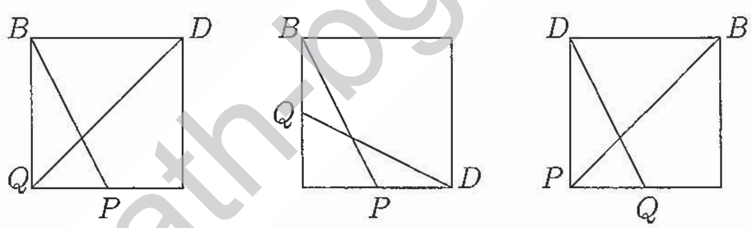




In a similar fashion the other 5 faces will produce a point to give 6 such points.

Also BP and DQ produce an internal point equidistant from the sides of the cube.
When we look at the front view, top view and side view of the cube (projections), we can see that this point of intersection is $\frac{1}{3}$ of a unit





from each face near the corner of the cube (from similar triangles with side ratio $2:1$ in each case). There are 8 corners and so 8 such points.

Thus the number of points of intersection found so far is $6 + 8 = 14$.

To show that there are no more points, consider AV. There are three possible lines that join B with the midpoints of the edges and do not lie on the surface of the cube. For BX and AV we get one of the 14 points that are described above. For BP and AV, consider their projections on the face ADG. Since their projections have only one common point at a corner of the square, they cannot intersect inside the cube. Checking all other pairs of lines in a similar fashion, we see that there are no more points of intersection except the ones discussed above, and the answer is therefore 14.

27. (Also J29 & I29)
We need to find the biggest sums we can get for a given number of terms. Some experiments:

1 term	$1 = 1$
2 terms	$2 = 1 + 1$
3 terms	$4 = 1 + 2 + 1$
4 terms	$6 = 1 + 2 + 2 + 1$
5 terms	$9 = 1 + 2 + 3 + 2 + 1$
6 terms	$12 = 1 + 2 + 3 + 3 + 2 + 1$

We can see that we get different formulas for odd or even numbers of terms.

For $2n$ terms, the maximum sum is $n(n+1)$.
For $2n+1$ terms , the maximum sum is $(n+1)^2$.

Now $44 \times 45 = 1980$ and $45 \times 45 = 2025$, so that 88 terms will not get us there, but 89 looks as though it should.

We can show it does, by starting with the biggest 88 term sum:

$$1980 = 1 + 2 + 3 + \ldots + 88 + 88 + 87 + \ldots + 3 + 2 + 1$$

This is 28 short of what we want, so put in a term of 28 next to one of the two 28s already there and we have a sum of 2008 with a minimum of 89 terms.

28. Now

$$3x^2 - 8y^2 + 3x^2y^2 = 2008 \Leftrightarrow 3x^2y^2 + 3x^2 - 8y^2 = 2008$$
$$3x^2y^2 + 3x^2 - 8y^2 - 8 = 2008 - 8 \Leftrightarrow 3x^2(y^2+1) - 8(y^2+1) = 2000$$
$$\Leftrightarrow (3x^2 - 8)(y^2+1) = 2000 = 2^4 \times 5^3.$$

The factors of 2000, arranged in corresponding pairs, are

(1,2000), (2,1000), (4,500), (5,400), (8,250),
(10,200), (16,125), (20,100), (25,80), (40,50).

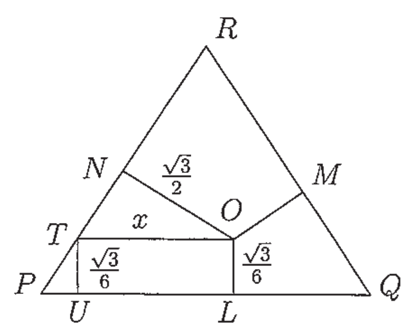
The only pair containing one number of the form $3x^2 - 8$ and the other of form $y^2 + 1$ is (40,50), where

$$3x^2 - 8 = 40 \text{ gives } 3x^2 = 48 \text{ and } x^2 = 16, x = 4.$$
$$y^2 + 1 = 50 \text{ gives } y^2 = 49 \text{ and } y = 7,$$

noting that x and y are positive integers.

Hence $xy = 4 \times 7 = 28$.

29. *Alternative 1*
Let the side of the equilateral triangle PQR be 2 units.
Then the altitude of the triangle is $\sqrt{3}$ units and the area of the triangle is $\sqrt{3}$ square units.
Draw OT parallel to QP to meet RP at T and draw TU perpendicular to PQ.



Now, by considering the areas of the triangles POQ, QOR and ROP, which combine to give the area of $\triangle PQR$, their altitudes combine to give that of $\triangle PQR$, which is $\sqrt{3}$.

So, $OL = \frac{\sqrt{3}}{6}$ and $ON = \frac{\sqrt{3}}{2}$.

Triangles ONT and TPU are 90°, 60° and 30° triangles.

So, $\dfrac{\sqrt{3}}{2} \div x = \dfrac{\sqrt{3}}{2}$ and $x = 1$.

So, the area of $\triangle ONT$ is $\frac{1}{2} \times \frac{\sqrt{3}}{2} \times \frac{1}{2} = \dfrac{\sqrt{3}}{8}$.

From $\triangle TPU$ we get $\dfrac{\sqrt{3}/6}{PU} = \sqrt{3}$, $PU = \dfrac{1}{6}$ and the area of $\triangle TPU = \frac{1}{2} \times \frac{1}{6} \times \frac{\sqrt{3}}{6} = \frac{\sqrt{3}}{72}$.

The area of rectangle $OTUL$ is $1 \times \frac{\sqrt{3}}{6} = \frac{\sqrt{3}}{6}$.

So, the area of $LONP$ is

$$\frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{6} + \frac{\sqrt{3}}{72} = \frac{22\sqrt{3}}{72} = \frac{11\sqrt{3}}{36}.$$

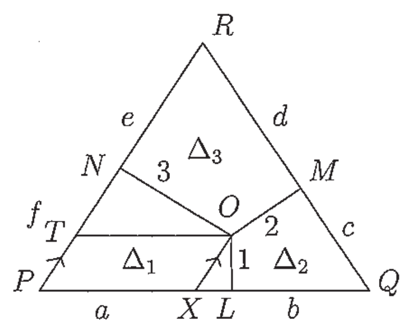
So, the ratio of the area of $LONP$ to the area of $\triangle PQR$ is

$$\frac{11\sqrt{3}}{36} : \sqrt{3} = \frac{11}{36}.$$

There are no common factors, so the sum of the numerator and denominator is $11 + 36 = 47$.

Alternative 2

Let Δ_1, Δ_2 and Δ_3 be the areas of the quadrilaterals.



$$\begin{aligned} 2\Delta_1 &= a + 3f \\ 2\Delta_2 &= b + 2c \\ 2\Delta_3 &= 2d + 3e \end{aligned}$$

So, to calculate $\dfrac{\Delta_1}{\Delta_1 + \Delta_2 + \Delta_3}$ we only have to calculate the lengths of a, b, c, d, e and f.

Construct $XO \parallel PR$ with X on PL. $XL = \tan 30° = \dfrac{1}{\sqrt{3}}$.

$\dfrac{ON}{TO} = \cos 30°$ so $TO = XP = \dfrac{3}{\cos 30°} = \dfrac{6}{\sqrt{3}}$. Hence $a = PL = \dfrac{7}{\sqrt{3}}$.

Similarly, let $OT \parallel PL$ with T on PN. $\dfrac{TN}{3} = \tan 30°$, $TN = \dfrac{3}{\sqrt{3}}$.

$\dfrac{1}{XO} = \cos 30° = \dfrac{\sqrt{3}}{2}$. So, $PT = XO = \dfrac{2}{\sqrt{3}}$ and $f = \dfrac{3}{\sqrt{3}} + \dfrac{2}{\sqrt{3}} = \dfrac{5}{\sqrt{3}}$.

Similarly we can obtain $b = \frac{5}{\sqrt{3}}$, $c = \frac{4}{\sqrt{3}}$, $d = \frac{8}{\sqrt{3}}$ and $e = \frac{7}{\sqrt{3}}$.

Hence

$$\begin{aligned}\frac{\Delta_1}{\Delta_1 + \Delta_2 + \Delta_3} &= \frac{a + 3f}{a + 3f + 2d + 3e + b + 2c} \\ &= \frac{(7+15)/\sqrt{3}}{(7+15+16+21+5+8)/\sqrt{3}} \\ &= \frac{22}{72} = \frac{11}{36}.\end{aligned}$$

The sum of the numerator and denominator is 47.

Alternative 3

Let h be the altitude of $\triangle PQR$.

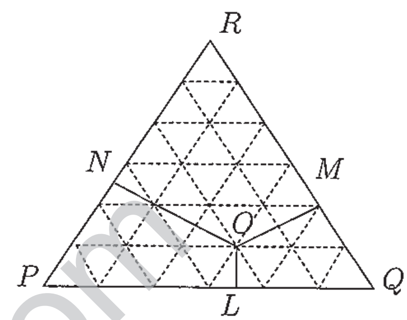
Considering the areas of the triangles OPQ, OPR and OQR, the sum of their altitudes is h, so $OL + OM + ON = 6OL$.

So $OL = \frac{h}{6}$, $OM = \frac{h}{3}$ and $ON = \frac{h}{2}$.

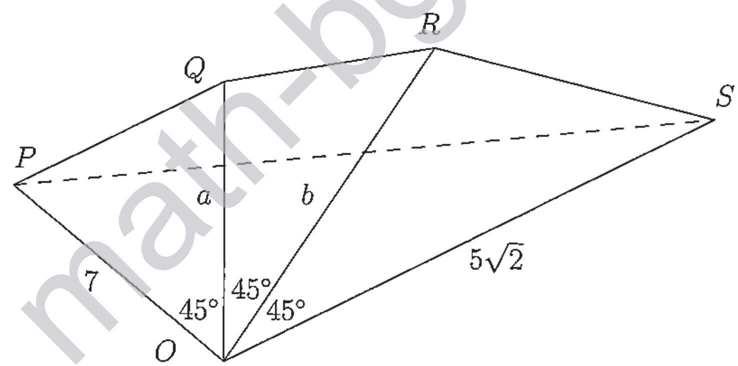
This places O on a triangular grid as shown.

So, counting the areas of the small triangles, we find that the area of $OLPN$ is $\frac{11}{36}$ of the area of $\triangle PQR$.

So the required number is $11 + 36 = 47$.



30. Consider a pentagon $OPQRS$ such that $OP = 7$, $OQ = a$, $OR = b$, $OS = 5\sqrt{2}$, and $\angle POQ = \angle QOR = \angle ROS = 45°$.



Then by the cosine theorem, $PQ = \sqrt{49 + a^2 - 7\sqrt{2}a}$, $QR = \sqrt{a^2 + b^2 - \sqrt{2}ab}$ and $RS = \sqrt{50 + b^2 - 10b}$.

By the triangle inequality, the smallest value that $PQ + QR + RS$ can have is PS, and by the cosine theorem it equals

$$\sqrt{OP^2 + OS^2 + \sqrt{2} \times OP \times OS} = \sqrt{49 + 50 + 70} = 13.$$

So the minimum value is 13.