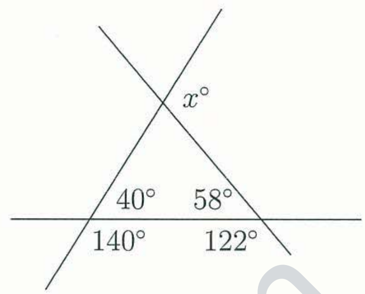
Solutions – Intermediate Division

1. (Also J6)
$(2000+9)+(2000-9)=4000+9-9=4000$,

hence (A).

2. (Also S2)

As we have straight lines, we can find the two additional angles as shown. Then angle x° is the exterior angle of the triangle and so $x=40+58=98$,



hence (E).

3. $\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}=\frac{1}{5}$,

hence (A).

4. (Also J10)
(A) is $\frac{1}{3}$, (B) is $\frac{2}{3}$, (C) is $\frac{1}{9}$, (D) is 0 and (E) is 1, so the largest is 1,

hence (E).

5. (Also J11)
We have

$$\begin{aligned}0.1\times0.2\times0.3\times0.4\times\square &= 0.12\\ 0.0024\times\square &= 0.12\\ \square &= \frac{0.12}{0.0024}=\frac{1200}{24}=50,\end{aligned}$$

hence (B).

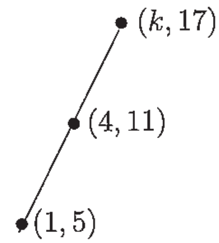
6. $3^k=9^{30}=(3^2)^{30}=3^{60}$, so $k=60$,

hence (D).

7. (Also S5)
$(x-y)-2(y-z)+3(z-x)=x-y-2y+2z+3z-3x=-2x-3y+5z$,

hence (A).

8. $17-11 = 11-5$, thus $k-4 = 4-1$ and $k = 7$,

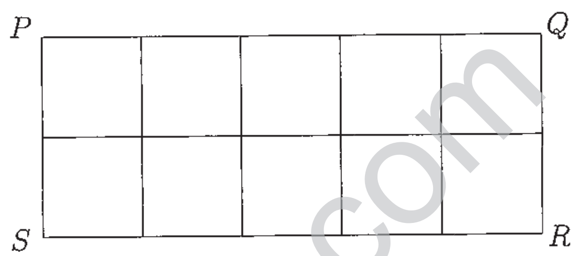


hence (E).

9. To maximise the number of books you must buy as few \$7 books as possible. Multiples of 5 end in 0 or 5, so consider: $86-0=86$, $86-7=79$, $86-14=72$, $86-21=65=13\times 5$.
So we can have three \$7 books and thirteen \$5 books, a total of 16,

hence (C).

10. (Also J12)
Let the side length of a square be s. Then the perimeter of the rectangle is $5s+2s+5s+2s=14s$.



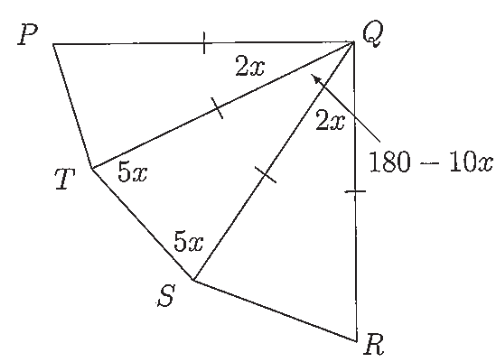
So $14s = 21$ and $s = \dfrac{21}{14} = \dfrac{3}{2} = 1.5\,\text{cm}$, so the perimeter of the square is $4\times 1.5 = 6\,\text{cm}$,

hence (C).

11. (Also S9)
There are 1000 students with 570 girls, so there are 430 boys.
One-quarter of the students travel by bus so 250 students travel by bus.
313 boys do not travel by bus so $430-313=117$ boys do take the bus.
So, the number of girls who travel by bus is $250-117=133$,

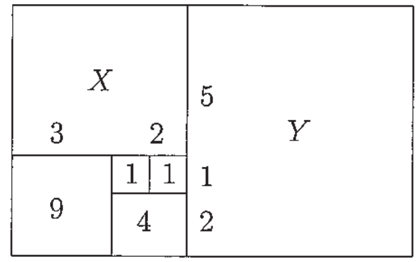
hence (E).

12. (Also J14)
With all angles in degrees, in the triangle QST, $\angle QST = 5x$ and then $\angle SQT = 180-10x$.
Now $\angle PQR = 90$, so $(180-10x)+4x=90$, $6x=90$ and $x=15$,



hence (D).

13. Filling in the lengths of the sides in the diagram, we find the side of the square X is 5 units and the side of the square Y is 8 units. So their areas are, respectively, $5^2 = 25$ and $8^2 = 64$,



hence (D).

14. $P : Q = 140 : 100 = 7 : 5$,

hence (E).

15. Noting the unit digits of the first five powers of 8 are 8, 4, 2, 6, 8, the units digit of powers of 8 cycles through 8, 2, 4, 6, 8.
$2009 = 4 \times 502 + 1$, so the last digit of 8^{2009} is 8 and the last digit of 6×8^{2009} is 8 as well,

hence (E).

16. Since the dice are twice as likely to show an even number as an odd, the probability of showing an even number is $\frac{2}{3}$ and the probability of showing an odd number is $\frac{1}{3}$. Now to obtain an odd product, both numbers must be odd, and the probability of this is $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$,

hence (A).

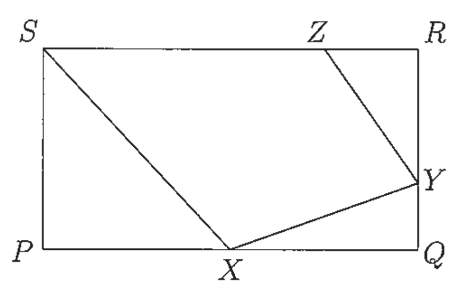
17. (Also S15)
Any arrangement in which 4 and 5 in some order are in the second and fourth positions, $x4x5x$ and $x5x4x$, is automatically an eyebrow. There are 2 ways of placing the 4 and 5, and then there are 6 ways of placing the other 3 numbers in the three positions, giving $2 \times 6 = 12$ eyebrows of this type.
However, it is also possible that 4 is at one end next to the 5, with 3 one place from the other end. There are 2 ways of choosing the end for the 4, and then the remaining two digits 1 and 2 can be placed in two ways, so there are $2 \times 2 = 4$ eyebrows of this type.
The total number of eyebrows is $12 + 4 = 16$,

hence (A).

18. $\triangle SPX$ is $\frac{1}{4}$ rectangle $PQRS$.

$\triangle XQY$ is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{12}$ rectangle $PQRS$.

$\triangle ZRY$ is $\frac{1}{2} \times \frac{1}{4} \times \frac{2}{3} = \frac{1}{12}$ rectangle $PQRS$.



So $XYZS$ is $1 - \frac{1}{4} - \frac{1}{12} - \frac{1}{12} = \frac{7}{12}$ rectangle $PQRS$,

hence (B).

19. *Alternative 1*

Let the fraction be t. Then

$$\begin{aligned} t-\frac{1}{t} &= \frac{9}{20} \\ \left(t-\frac{1}{t}\right)^2 &= \frac{81}{400} \\ t^2-2+\frac{1}{t^2} &= \frac{81}{400} \\ t^2+2+\frac{1}{t^2} &= \frac{81}{400}+4 \\ \left(t+\frac{1}{t}\right)^2 &= \frac{1681}{400} \\ t+\frac{1}{t} &= \frac{41}{20} \end{aligned}$$

hence (D).

Alternative 2

Let the fraction be x. Then

$$\begin{aligned} x-\frac{1}{x} &= \frac{9}{20} \\ 20x^2-20 &= 9x \\ 20x^2-9x-20 &= 0 \\ (5x+4)(4x-5) &= 0 \end{aligned}$$

So $x=-\frac{4}{5}$ or $\frac{5}{4}$, but $x>0$, so $x+\frac{1}{x}=\frac{5}{4}+\frac{4}{5}=\frac{41}{20}$,

hence (D).

20. From figure 1, we find the volume of the air in the bottle is

$$\begin{aligned} V &= \pi\times\left(\frac{5}{2}\right)^2\times 2+\pi\times\left(\frac{1}{2}\right)^2\times 3 \\ &= \frac{25\pi}{2}+\frac{3\pi}{4} \\ &= \frac{53\pi}{4} \end{aligned}$$

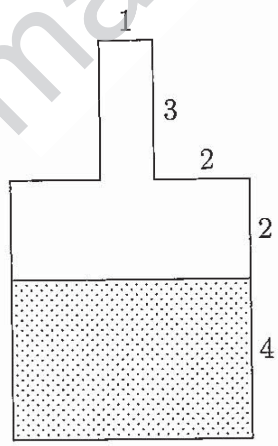


figure 1

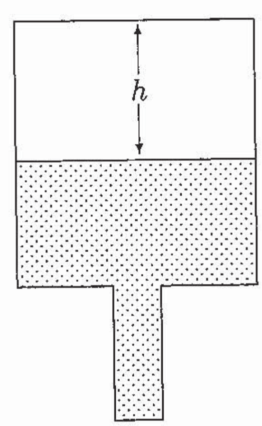


figure 2

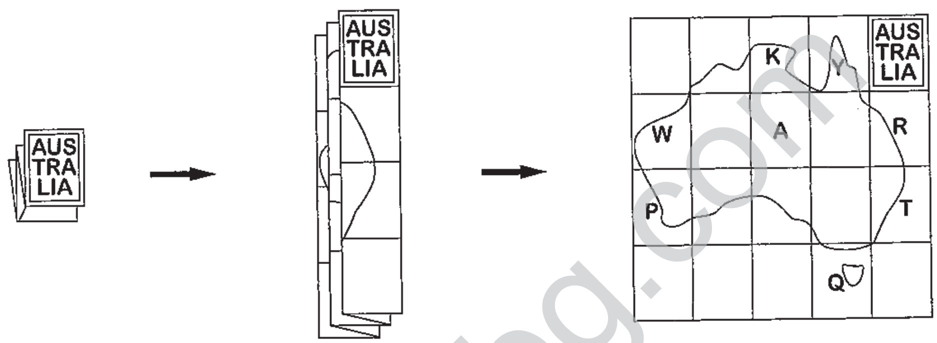
This remains constant when the bottle is upturned, so

$$\begin{aligned} \frac{53\pi}{4} &= \pi \times \left(\frac{5}{2}\right)^2 \times h \\ &= \frac{25\pi h}{4} \end{aligned}$$

So, $\dfrac{25\pi h}{4} = \dfrac{53\pi}{4}$ and $h = \dfrac{53}{25} = 2\dfrac{3}{25}$,

hence (E).

21. (Also J23 & S20)
After refolding along vertical folds, the four panels are stacked, from top to bottom, YK, RAW, TP, Q.



After folding the horizontal folds, the second and fourth will be reversed giving YK, WAR, TP, Q,

hence (E).

22. (Also J25 & S21)
Any palindromic number $xyyx$ can be written as $1000x + 100y + 10y + x = 1001x + 110y$, where x and y are integers and $1 \leq x \leq 9$ and $0 \leq y \leq 9$.

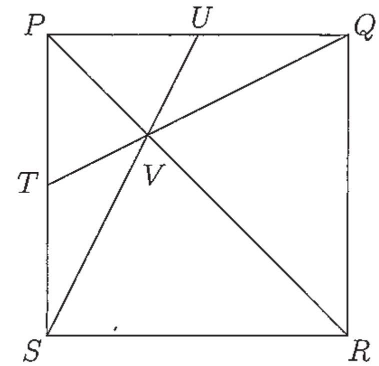
Now $1001 = 7 \times 143$, so 1001 and every multiple of it is divisible by 7. There are nine such multiples 1001, 2002, 3003, …, 9009.

110 is not divisible by 7, so $110y$ is not divisible by 7 unless y is divisible by 7, and this occurs when $y = 0$ (already dealt with above) or $y = 7$. This gives another nine palindromes, 1771, 2772, …, 9779.

So there are $9 + 9 = 18$ such palindromes,

hence (D).

23. From the symmetry, V lies on the diagonal PR.
Triangles PUV, PVT, TVS and UVQ are equal in area.
Triangles PUV, PVT and TVS form $\triangle PSU$ which is $\frac{1}{4}$ of the square $PQRS$.
So $\triangle PUV = \triangle PVT = \triangle TVS = \triangle UVQ = \dfrac{1}{12}$ area of square $PQRS$.



Then the area of quadrilateral $SVQR$ is $1 - \dfrac{4}{12} = \dfrac{2}{3}$ area of the square $PQRS$,

hence (C).

24. (Also S23)
A is either loyal or a traitor.
Assume that A is loyal. B's claim shows B is loyal, F's claim shows F is loyal, D's claim shows D is a traitor. A's claim shows E is a traitor. C's claim is now impossible as B and F are loyal; C would either claim both were loyal or both were traitors.
Hence A is a traitor. So by F's claim F is a traitor. A says E is a traitor so E is loyal. Hence by E's claim, D is a traitor. By D's claim C is a traitor. Finally by C's claim B is loyal.
So, B and E are loyal and A, C, D and F are traitors,

hence (D).

25. Let the integers be $n-3, n-2, n-1, n, n+1, n+2$ and $n+3$. Then
$(n-3)^2 + (n-2)^2 + (n-1)^2 + n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2$
$= n^2 - 6n + 9 + n^2 - 4n + 4 + n^2 - 2n + 1 + n^2 + n^2 + 2n + 1 + n^2 + 4n + 4 + n^2 + 6n + 9$
$= 7n^2 + 28$.

The last digit of n^2 is 0, 1, 4, 5, 6 or 9.
So the last digit of $7n^2$ is 0, 7, 8, 5, 2 or 3.
Then the last digit of $7n^2 + 28$ is 8, 5, 6, 3, 0 or 1, so it cannot be 7,

hence (D).

26. The smallest number which has a remainder of 1 when divided by 2, a remainder of 2 when divided by 3, a remainder of 3 when divided by 4, a remainder of 4 when divided by 5, a remainder of 5 when divided by 6 and a remainder of 6 when divided by 7 is one less than the smallest number divisible by 2, 3, 4, 5, 6 and 7. So it is one less than the LCM of 2, 3, 4, 5, 6 and 7, which is $2 \times 3 \times 2 \times 5 \times 7 - 1 = 420 - 1 = 419$.

27. (Also J28 & S27)

The seven smallest ascending 3-digit numbers are
$n = 123, 124, 125, 126, 127, 128$ and 129.

$6 \times 123 = 738 \quad 6 \times 127 = 762$
$6 \times 124 = 744 \quad 6 \times 128 = 768$
$6 \times 125 = 750 \quad 6 \times 129 = 774$
$6 \times 126 = 756$

In none of these cases is $6n$ an ascending number.

Now, $6n$ must end with a 0, 2, 4, 6 or 8 and the sum of its digits must be divisible by 3.
124, 134 and 234 are the only ascending 3-digit numbers ending with 4 and $6n = 744, 804, 1404$ in these cases, none ascending.
Hence n ends in a 6 or an 8.

Consider those ending in 6.

n	$6n$	n	$6n$
136	816	146	876
156	936	236	1416
246	1476	256	1536
346	2076	356	2136
456	2736		

None of the $6n$ are ascending.

Consider then n ending with an 8, so is $ab8$ with $a < b < 8$. Then the carry into the tens digit of the product $6 \times n$ is 4. The next largest digit in n we can consider is $b = 7$ which gives $42 + 4$, so we get 6 in the tens digit of $6n$ and a carry of 4 to the hundreds digit of $6n$. This means the units digit of $6a$, must be less than 1 (otherwise $6n$ would not be ascending). Hence the first digit of n is 5, $6n$ is 3468 and n is 578.

28. (Also J30)

Alternative 1

Suppose Merlin starts with a rabbits and leaves b rabbits at each house. Then

place	number of rabbits
arrives house 1	$2a$
leaves house 1	$2a - b$
arrives house 2	$2(2a - b) = 4a - 2b$
leaves house 2	$4a - 3b$
arrives house 3	$2(4a - 3b) = 8a - 6b$
leaves house 3	$8a - 7b$
arrives house 4	$2(8a - 7b) = 16a - 14b$
leaves house 4	$16a - 15b$
arrives house 5	$2(16a - 15b) = 32a - 30b$
leaves house 5	$32a - 31b$

So, if he leaves the last house with no rabbits, then $32a - 31b = 0$ and $32a = 31b$. The smallest values of a and b are 31 and 32 respectively, so the minimum number of rabbits he could have at the start is 31.

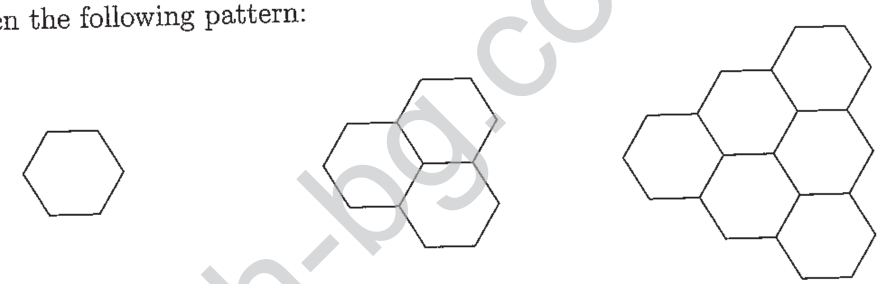
Alternative 2
Assume that Merlin starts with x rabbits and leaves y rabbits at each of the houses. Then

$$\begin{aligned}((((2x-y)2-y)2-y)2-y)2-y &= 0\\ (((4x-3y)2-y)2-y)2-y &= 0\\ ((8x-7y)2-y)2-y &= 0\\ (16x-15y)2-y &= 0\\ 32x-31y &= 0\\ x &= \frac{31y}{32}\end{aligned}$$

The smallest value of y to make x an integer is $y = 32$. So the minimum number of rabbits he has at the start is 31.

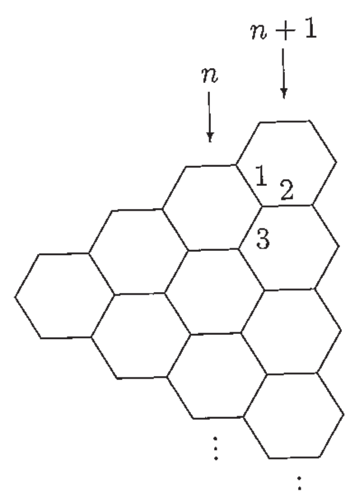
Generalisation
If there are n houses, the minimum number he could have at the start is $2^n - 1$.

29. *Alternative 1*
Given the following pattern:



pattern 1 pattern 2 pattern 3

We can see that pattern 1 has 1 hexagon, pattern 2 has 2 hexagons on its right-hand side, pattern 3 has 3 hexagons on the right-hand side so we can see that pattern n will have n hexagons on the right-hand side, pattern $n+1$ will have $n+1$, and so on.



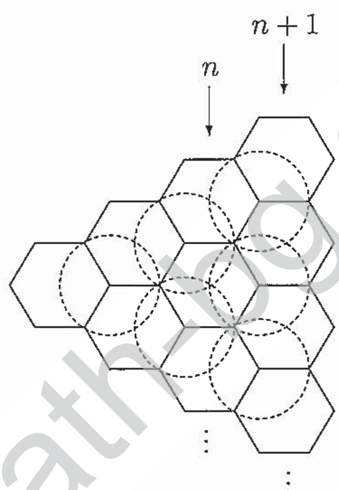
So, to create pattern $n+1$ from pattern n, we are adding $n+1$ hexagons to the right-hand side. This adds $6(n+1)$ extra line segments, with some counted twice. This is 3 line segments for every one of the hexagons in the right-hand side of pattern n as shown in the diagram.
So, the increase in the number of line segments when we create pattern $n+1$ from pattern n is $6(n+1) - 3n = 3n+6$.
So we can build the following sequence:

pattern number	1		2		3		4		5		6
line segment increase		9		12		15		18		21	
number of line segments	6		15		27		42		60		81

pattern number		7		8		9		10		11
line segment increase	24		27		30		33		36	
number of line segments		105		132		162		195		231

So, the number of line segments in pattern 11 is 231.

Alternative 2
Let T_n be the nth triangular number. By inspection, the number of hexagons in pattern n is T_n.



So, the number of sides of the hexagons is $6T_n = \dfrac{6n(n+1)}{2}$, where the sides of the hexagons can be grouped in threes, and the number of sides which are counted twice is $3T_{n-1}$.
Hence the number of sides is
$S_n = 6T_n - T_{n-1} = \dfrac{6n(n+1)}{2} - \dfrac{3n(n-1)}{2} = \dfrac{3n(n+3)}{2}$.
In this particular case we want S_{11} which is $\dfrac{33 \times 14}{2} = 231$.

30. (Also S29)

Alternative 1

Without any loss of generality, assume the track is circular, and put in the stations C_1 to C_5 with C_1 at the top. There are then five 72° gaps.

Then add the B stations. As there are 4 of these, there must remain a clear gap between two C stations. Assume that this is the C_1C_5 gap. Put B_1 in the C_1C_2 gap $x°$ from C_1.

Then $x \le 18$ or B_4 would lie in the C_1C_5 gap and through symmetry, would put the clear gap elsewhere.

Then place the A stations. Clearly, A_1 must go in the 72° C_1C_5 gap. Let it make an angle $y°$ with C_1.

Then, if $y < 24$, A_3 lies between B_4 and C_4, and using the fact that the angles between the C stations are 72°, the angles between the B stations are 90° and the angles between the A stations are 120°, we can fill in all the angles as shown in figure 1.

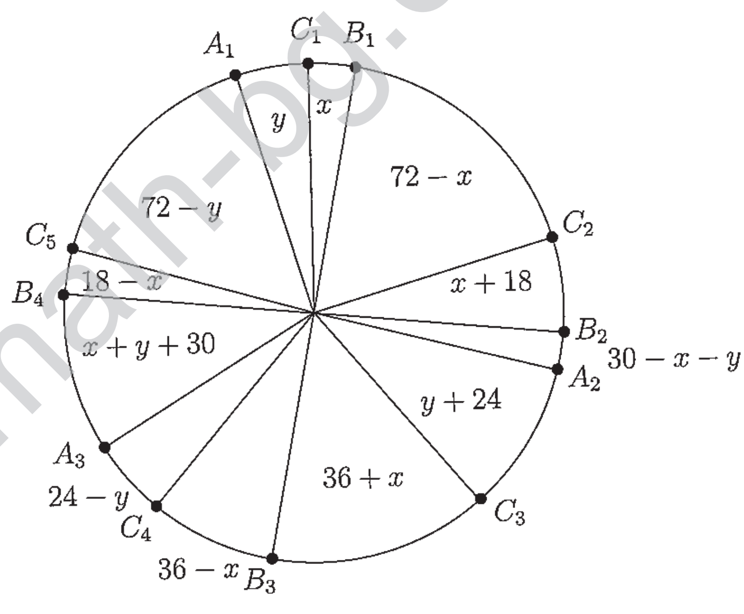


figure 1

If, however, from $\angle A_2OB_2$, $x + y > 30$, then A_2 is between B_2 and C_2 and we obtain the angles as in figure 2.

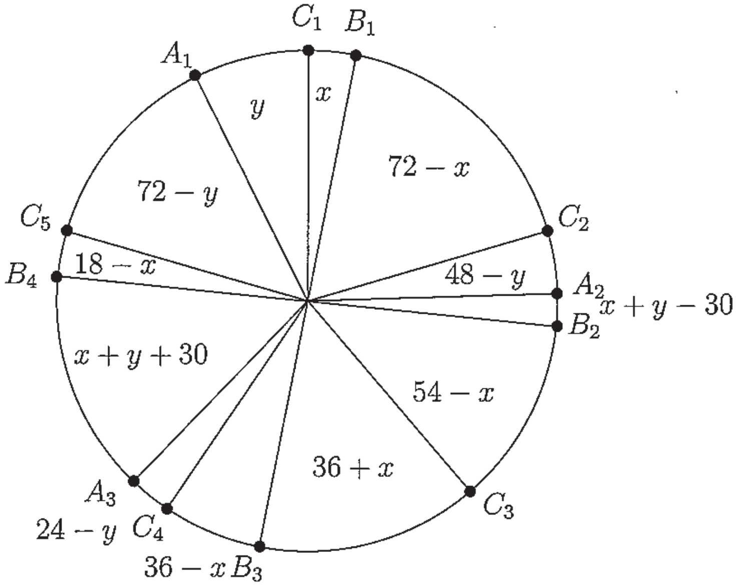


figure 2

In both cases, it is clear that the largest angles are $72 - x$, $72 - y$ and $x + y + 30$. So, the smallest maximum will be when all three are equal:

$$72 - y = 72 - x = x + y + 30$$

This gives $x = y = 14$ and the smallest of the largest gaps is $58°$.

For $24 \leq y \leq 48$, then A_3 is between C_4 and B_3 as in figure 3.

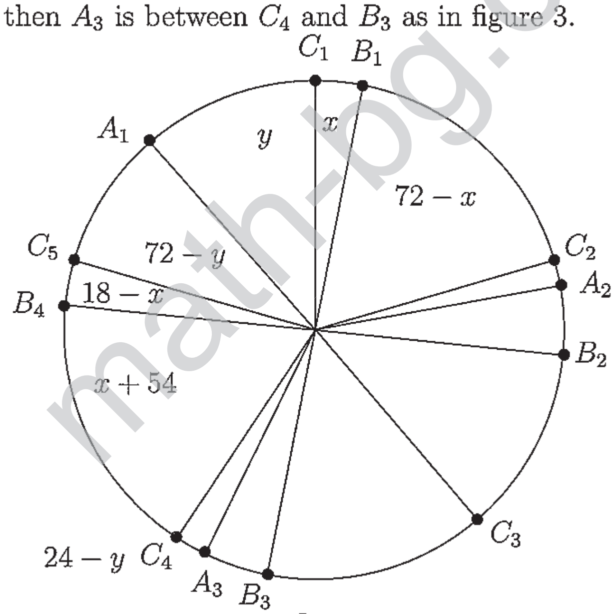


figure 3

This forces two gaps of $x + 54$ and $72 - x$, where one or both must be greater than 58.

When $48 < y < 72$, then A_2 lies between B_1 and C_2 and A_3 lies between B_3 and C_3 as in figure 4.

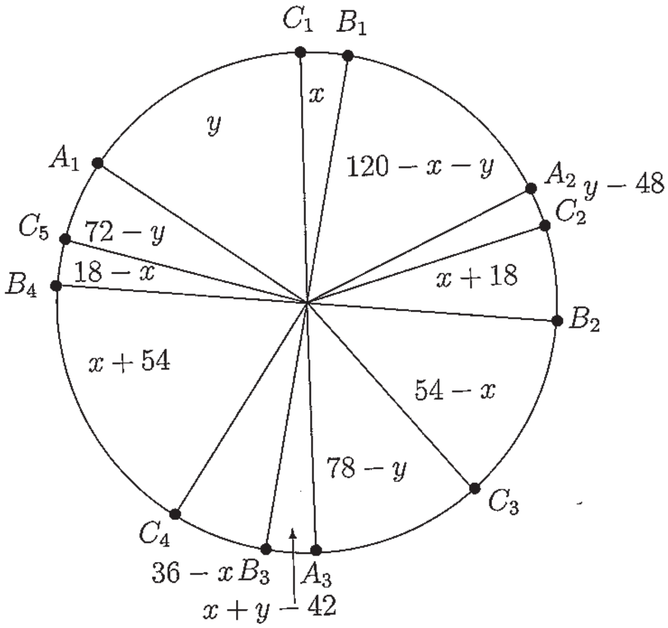


figure 4

Then the largest angles are y, $120 - x - y$ and $x + 54$.
Then, when $y = 120 - x - y = x + 54$, we get $y = 58$, $x = 4$ and the smallest of the largest angles is $y = 54 + x = 120 - x - y = 58$, the same as for figure 1.

The minimum of the largest distances between stations is then
$\frac{58}{360} \times 1080 = 174\,\text{km}$.

Alternative 2
The maximum spacing must be at least 90 km, as there are 12 stations in all. However, the station spacing is severely constrained, so the actual maximum will be well in excess of this.

The maximum spacing must be at most 216 km, as this is the spacing of the C stations. Adding in the B stations will not reduce this, as there will always be a pair of C stations with no B station in between.

First, a 'reasonably good' solution:
Fix the C stations as a reference set, and position the A and B stations relative to them. Initially, align the B and C stations at km 0; the largest gap is 216 km, at either side of km 0. Next, align the A and B stations at km 540; the other A stations will both be in the previous longest sections, reducing them to 180 km each. The pattern and spacings are:

$$BC\ 180\ A\ 36\ C\ 54\ B\ 162\ C\ 108\ AB\ 108\ C\ 162\ B\ 54\ C\ 36\ A\ 180\ BC.$$

This is already close to optimum, as we have 2 gaps of 180 km and 2 of 162 km.

Now, improve the fit by breaking the symmetry: alter the A and B starting points to decrease both of the 180 km gaps. Move the A system forward by x km and the B system forward by $2x$ km, reducing both 180 km gaps by x km each. As the second of the 162 km gaps is now increased by $2x$ km, x cannot exceed 6. Using $x = 6$, the new pattern is

$$C\ 12\ B\ 174\ A\ 30\ C\ 66\ B\ 150\ C\ 114\ A\ 6\ B\ 96\ C\ 174\ B\ 42\ C\ 42\ A\ 174\ C.$$

Note that we now have 3 gaps of 174 km each and the next largest is 150 km. Further alterations will result in at least one of the 174 km gaps increasing in length, so we have an optimum: 174 km.